Gompertz Lomax distribution: Estimations and properties

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Summary

This research investigates four distinct techniques for estimating parameters of the Gompertz Lomax (GL) distribution. It employs both full sample data and Maximum Likelihood Estimation (MLE) under two forms of censoring, denoted as type-I and type-II. The performance of these estimators is evaluated by analyzing their squared biases and variances using Monte Carlo simulation methods. Furthermore, the study offers a detailed examination of the distribution's ordinary moments and its quantile function.

Keywords: quintile; Gompertz G; Lomax; Progressive Type-II Censoring; moments

1.Introduction

The Lomax distribution—also known as the shifted Pareto or Pareto Type II—was originally introduced by Lomax in 1954 [11] to analyze patterns in business failure data. Since then, it has found widespread use across diverse domains, including economics (e.g., income and wealth disparity), engineering, healthcare, biological research, and survival analysis.

This distribution can be described through multiple formulations. It represents a specific instance of the Pearson Type VI distribution and can also be interpreted as a hybrid of the exponential and gamma distributions. Despite its versatility, the Lomax model is not well-suited for datasets exhibiting non-monotonic hazard functions, such as those with bathtub-shaped or inverted bathtub failure rates—patterns frequently observed in biological and survival studies.

Mathematically, the probability density function (PDF) of the Lomax distribution is defined using two parameters: the shape parameter α and the scale parameter β. Is given by:

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta}(1+\frac{x}{\beta})^{-\alpha-1} \quad \dots\dots\dots(1)$$

And CDF given by:

$$F(x)=1-(1+\frac{x}{\beta})^{-\alpha} \quad \dots\dots\dots\dots\dots\dots(2)$$

$$where\ x>0,\alpha,\beta>0$$

There is a growing demand for generalized versions of the Lomax distribution across numerous practical disciplines. This study expands on earlier investigations within this field.

M. H. Tahir (2010) [16] introduced a new model called the Weibull-Lomax distribution using the Weibull-G generator. Artur J. Lemonte (2011) [15] proposed A new five-parameter continuous distribution, the so-called McDonald Lomax distribution, that extends the Lomax distribution using the McDonald-G distribution. Tarek M. Shams (2013) [26] proposed an extension of the Lomax distribution based on the family of Kumaraswamy generalized ("Kw-G"). Hamdy M. Salem (2014) [6] proposed the estimation of parameters for the Exponentiated Lomax Distribution Different estimation methods. A. H. El-Bassiouny (2015) [2] introduced A new distribution called Exponential Lomax Distribution. N. M. Kilany (2016) [21] proposed Weighted Lomax distribution proposed. Pelumi E. Oguntunde (2017) [23] introduced A New Generalization of the Lomax Distribution. Chookait Pudprommarat (2018) [5] proposed a new mixture Lomax distribution for positive continuous random variable. Sharqa Hashmi (2018) [25] introduced Weibull-Lomax (T-X) distribution. Mohamed Aboraya (2021) [18] introduced a new four-parameter lifetime probability distribution called the Marshall-Olkin Lehmann Lomax distribution. Ibrahim Sule. (2021) [8] introduced a new four-parameter lifetime distribution called the Topp Leone exponentiated Lomax distribution. M. Masoom Ali (2021) [17] defined and studied A new three-parameter lifetime model called the odd Burr Lomax (OBLo). Refah Alotaibi (2021) [24] introduced a new extension of the traditional Lomax distribution. Nagarjuna Vasili B. V. (2022) [22] proposed generalizations or extensions of the Lomax distribution, an attempt is made to create a hybrid distribution mixing the functionalities of the Nadarajah–Haghighi and Lomax distributions. Laxmi Prasad Sapkota (2023) [14] introduced a new family called Lomax G-Family. Aliyu Ismail Ishaq (2025) [3] introduced the log-Lomax distribution, a more versatile probabilistic model for capturing various statistical properties. Hemani Sharma (2025) [7] studied the maximum likelihood (ML) estimation and the Bayesian approach for parameter estimation of the Lomax distribution

2. Gompertz-G family

Alzaatreh et al. (2013) [2] introduced a comprehensive approach for constructing distribution families. Their method involves utilizing a probability density function (PDF) denoted as r(.) for a continuous random variable. By applying a function $W(\mathrm{G}(x))$, which adheres to specified conditions, they systematically create the T-X family. The cumulative distribution function (CDF) of this T-X family is explicitly defined as part of their innovative methodology.

$$\mathrm{F(x)} = \int_a^{w[\mathrm{G}(x)]} \mathrm{r(t)}dt \tag{3}$$

$W\left[G\left(x\right)\right]$ is differentiable and monotonically non-decreasing, and $W\left[G\left(x\right)\right] \rightarrow a$ as $x \rightarrow -\infty$ *and* $W\left[G\left(x\right)\right] \rightarrow b$ *as* $x \rightarrow \infty$. The pdf corresponding to (1)is given by

$$f(x) = \left\{\frac{d}{dx} w[G(x)]\right\} r\{w[G(x)]\} \dots\dots\dots\dots.(4)$$

Morad Alizadeh (2017) [20] introduction *Gompertz-G as following:*

Setting $W[G(x)] = -\ln[1 - G(x)]$ and $r(t) = \lambda e^{\theta t} exp\{-\frac{\lambda}{\theta}(e^{\theta t} - 1)\}$,

$t > 0$, we define the cdf of the *Gompertz-G* (Go-G for short) family by:

$$G(x) = \int_0^{-ln[1-F(x)]} \lambda e^{\theta t} e^{-\frac{\lambda}{\gamma}(e^{\theta t}-1)} dt = 1 - e^{\frac{\lambda}{\theta}\{1-[1-F(x)]^{-\theta}\}} \dots..(5)$$

we define the pdf of the *Gompertz-G* family by:

$$g(x) = \lambda f(x;v)[1 - F(x;v)]^{-\theta-1} e^{\frac{\lambda}{\theta}\{1-[1-F(x;v)]^{-\theta}\}}. \dots\dots\dots(6)$$

We obtained the PDF and the CDF of Gompertz Lomax (GL) by putting (1) and (2) in (5) and (6) as:

$$g(x) = \lambda\frac{\alpha}{\beta}(1 + \frac{x}{\alpha})^{-\alpha-1}[(1 + \frac{x}{\beta})^{-\alpha}]^{-\theta-1} exp\{\frac{\lambda}{\theta}(1 - (1 + \frac{x}{\alpha})^{\theta\alpha})\} \dots\dots\dots..(7)$$

$$G(x) = 1 - exp\{\frac{\lambda}{\theta}(1 - (1 + \frac{x}{\alpha})^{\theta\alpha})\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (8)$$

$$s(x) = 1 - G(x) = 1 - 1 + exp\{\frac{\lambda}{\theta}\left(1 - \left(1 + \frac{x}{\alpha}\right)^{\theta\alpha}\right)\}$$

$$\therefore s(x) = exp\{\frac{\lambda}{\theta}\left(1 - \left(1 + \frac{x}{\alpha}\right)^{\theta\alpha}\right)\} \dots\dots\dots\dots\dots\dots\dots\dots\dots..(9)$$

$$h(x) = \frac{g(x)}{s(x)} = \lambda\frac{\alpha}{\beta}(1 + \frac{x}{\alpha})^{-\alpha-1}[(1 + \frac{x}{\beta})^{-\alpha}]^{-\theta-1} \dots\dots\dots\dots..(10)$$

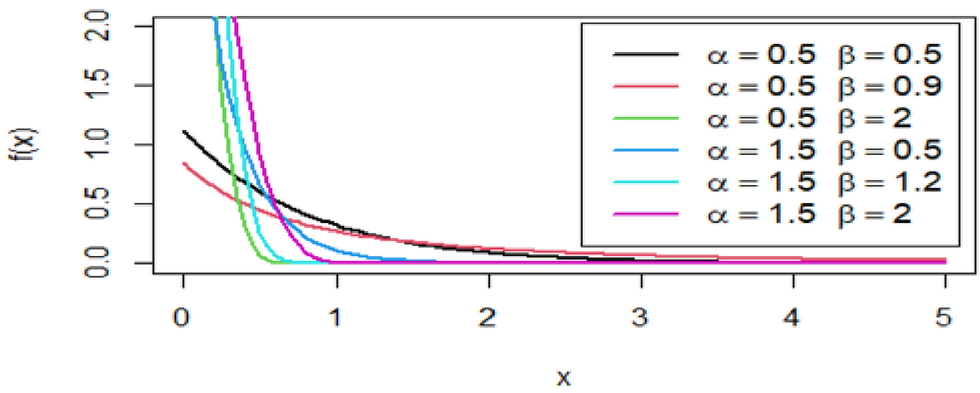

Figure 1. PDF of GL distribution at $\lambda = 1.5. \theta = 2.5$

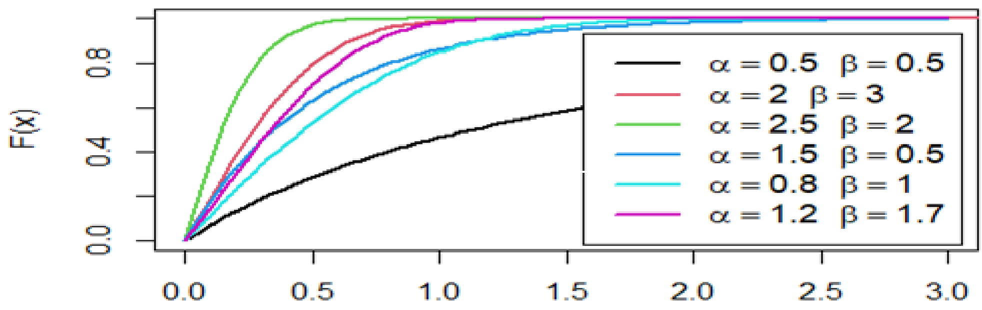

Figure 2. CDF of GL distribution $\lambda = 1.5. \theta = 2.5$

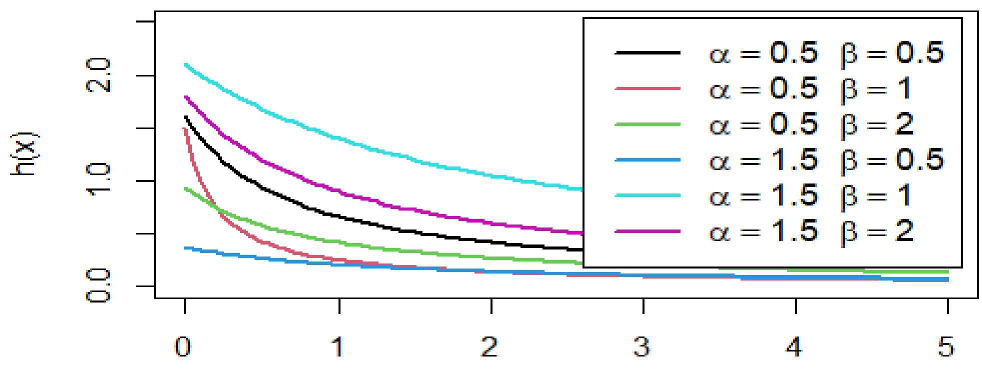

Figure 3. Hazard of GL distribution $\lambda = 1.5. \theta = 2.5$

3- the Generating function:

The generating function (x_u) is the solution of the equation:

$x_u = G(u)$

Where u is the uniform random variable u (0,1)

The generating function (x_u) of the **GL** distribution is given by:

$$x_p = \beta\left[1 - \frac{\lambda}{\theta} ln(1-p)\right]^{\frac{1}{\alpha\lambda}} - a \dots\dots\dots\dots\dots\dots\dots (11)$$

The Bowley Skewness measure Drapella [1] and the Moor's Kurtosis measure Balanda [4,19] are defined by

$$Bsk = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}}$$

$$Mkur = \frac{Q_{0.875} - 2Q_{0.625} - 2Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}$$

table.1 Quantile

α, β, λ, Υ	0.25	0.5	0.7	0.125	0.375	0.625	0.875	Bskewne	Mkurtosis
0.5,0.8,0.7,1.2	0.27769	0.63971	1.2586	0.14006	0.44048	0.89657	1.87748	0.25686	1.28132554
0.7, 0.8, 1, 0.5	1.30172	1.6565	2.26301	1.16684	1.46125	1.90823	2.86951	0.25172	1.25569891
1, 1.2, 1.5, 0.8	5.45894	6.81049	9.12098	4.9451	6.06668	7.76943	11.4315	0.95894	4.78361512
0.2, 0.5, 1.2, 1.5	0.20149	0.42675	0.81183	0.11585	0.30278	0.58657	1.19691	0.15982	0.79726922
0.5, 0.7, 1.7, 1.8	0.16256	0.29772	0.52876	0.11118	0.22333	0.39361	0.75981	0.09589	0.47836141
0.7, 0.9, 0.5, 0.2	0.0177	0.18806	0.4793	(-)0.0470	0.09431	0.30894	0.77054	0.12087	
0.5, 0.5, 0.3, 0.2	3.63639	4.26712	5.34535	3.3966	3.92001	4.71462	6.42358	0.44751	2.23235366

The above table shows that both skewness and kurtosis decrease with increasing α, β, λ, and θ.

4. Moments:

Moments [13] are fundamental in characterizing a probability distribution, providing insights into its central tendency, dispersion, and shape (skewness and kurtosis). The n^{th} moment about the origin, $E[X^n]$, for a continuous random variable X with PDF $f(x)$ is given by:

$$E[X^n] = \int_0^{\infty} x^n g(x)dx$$

$$= \int_0^{\infty} x^n \lambda \frac{\alpha}{\beta}(1+\frac{x}{\alpha})^{-\alpha-1}[(1+\frac{x}{\beta})^{-\alpha}]^{-\theta-1} exp\{\frac{\lambda}{\theta}(1-(1+\frac{x}{\alpha})^{\theta\alpha})\}dx..(12)$$

Table 2. the values of moments for different values of λ, α, β and γ

moments	.5, .5, .7, 1.2	.7, .8, 1, 1.2	1, 1.2, 1.5, .8	0.2, 0.5, 1.2, 2.5
μ	0.913805	0.705177	0.546658	0.265797
μ_2	1.733382	0.870254	0.491887	0.09328
μ_3	4.446708	1.615761	0.682633	0.050008
μ_4	13.23356	3.820679	1.265633	0.035881
σ^2	0.898342	0.372979	0.193052	0.022632
CV	1.037211	0.866053	0.803751	0.565997
sk	1.433914	2.089877	2.389357	3.87205
kur	1.426969	5.034973	7.384412	11.2129

5. Methods of Estimation:

In this section we estimate the parameters of the GL distribution by 5 different methods using complete sample technique. These methods are: maximum likelihood (MLE), Least-squares (LS), weighted least squares and percentile-based estimation. The performance of all methods is studied by the R software.

5.1 -Maximum likelihood estimation (MLE):

The MLE method is a general method and its estimators have some optimum properties such as consistency, asymptotic efficiency and invariance property.

Let $\{x_1, x_2, \ldots. x_n\}$ be a random sample from GL population with PDF g(x) given in (5) with unknown parameters β, λ and α and the log-likelihood function is 1(θ, β, λ, α) then the MLE estimates of θ, λ and α are the simultaneously solution of the following equations:

$$\frac{\partial \ell(\beta,\alpha,\lambda,\theta)}{\partial \alpha} = 0, \frac{\partial \ell(\beta,\alpha,\lambda,\theta)}{\partial \beta} = 0, \frac{\partial \ell(\beta,\alpha,\lambda,\theta)}{\partial \theta} = 0, \frac{\partial \ell(\beta,\alpha,\lambda,\theta)}{\partial \lambda} = 0$$

$$ln = n \log \lambda + \sum \left[\log\left(\frac{\alpha}{\beta}\right) - (\alpha+1)\log\left(1+\frac{t}{\beta}\right) \right] + \alpha(\theta+1)\log\left(1+\frac{t}{\beta}\right) + \left(\frac{\lambda}{\theta}\right)\sum \log\left\{1-\left(1+\frac{t}{\beta}\right)^{\theta\alpha}\right\}$$

$$\frac{\partial ln}{\partial \lambda} = \frac{n}{\lambda} + \frac{1}{\theta}\Sigma \log\left\{1-\left(1+\frac{t}{\beta}\right)^{\alpha\theta}\right\}$$

Which gives the MLE estimates $(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \lambda)$. *These equations are solved numerically using R software.*

$$\frac{\partial ln}{\partial \lambda} = \frac{n}{\lambda} + \frac{1}{\theta} \Sigma\, log \left\{ 1 - \left(1 + \frac{t}{\beta}\right)^{\alpha\theta} \right\}$$

$$\frac{\partial ln}{\partial \theta} = \Sigma\, \alpha log \left(1 + \frac{t}{\beta}\right) \; - \frac{\lambda}{\theta} \Sigma \frac{\alpha \left(1 + \frac{t}{\beta}\right)^{\alpha\theta} log \left(1 + \frac{t}{\beta}\right)}{1 - \left(1 + \frac{t}{\beta}\right)^{\alpha\theta}} - \frac{\lambda}{\theta^2} \Sigma \log \{1 - \left(1 + \frac{t}{\beta}\right)^{\alpha\theta} \}$$

$$\frac{\partial ln}{\partial \beta} = -\frac{n}{\beta} + \frac{(\alpha+1)t}{\beta^2 + \beta t} - \frac{(\alpha+1)t}{\beta^2 + \beta t} - \frac{\lambda\alpha \left(1 + \frac{t}{\beta}\right)^{\alpha\theta - 1} t}{\beta^2 [1 - \left(1 + \frac{t}{\beta}\right)^{\alpha\theta}]}$$

$$\frac{\partial ln}{\partial \alpha} = \frac{n}{\alpha} - log \left(1 + \frac{t}{\beta}\right) \; + (\alpha+1) \log \left(1 + \frac{t}{\beta}\right) \; - \lambda \frac{\left(1 + \frac{t}{\beta}\right)^{\alpha\theta} log \left(1 + \frac{t}{\beta}\right)}{1 - \left(1 + \frac{t}{\beta}\right)^{\alpha\theta}}$$

5.2 -Method of Ordinary Least Squares (L):

The best estimates according to LS method [10] are those which minimize the following quantity:

$$Q_1 = \sum_{i=1}^{n} (G(x_{(i)} - \frac{i}{n+1}))^2$$

With respect to β, θ, λ and α.

where. $x_{(i)}$ *is the Ith orders statistic of GL*

5.3 -Method of Weighted Least Squares (WLS):

The WLS estimators of β, θ, λ and α of GL distribution can be obtained by minimizing the quantity:

$$Q_2 = \sum_{i=1}^{n} \frac{(\mathrm{n}+1)^2 (n+2)}{n-i+1} (G(x_{(i)} - \frac{i}{n+1}))^2$$

With respect to β, θ, λ and α

5.4 -Method of Percentile Estimation (PCE) :

This method introduced by Kao [9,11] the PCEs estimators of β, θ, λ and α of GL distribution can be obtained by minimizing the quantity:

$$Q = \sum_{i=1}^{n} [x_{(i)} - a(\frac{1-(1-\sigma)(i/n+1)}{1-(i/n+1)})^{\frac{1}{\sigma^2}}]$$

where. $x_{(i)}$ *is the I th orders statistic of GL*

5.5 -Simulation Study

This section presents a Monte Carlo simulation designed to evaluate how various estimators perform when applied to the unknown parameters of the GTL distribution. Using the R programming environment, numerical comparisons are made based on the mean squared error (MSE) criterion. For the analysis, 1,000 samples are generated from the GL distribution, with sample sizes set at 25, 50, 75, and 100.

Tables 3 and 4 present the mean estimates (AVEs) and mean squared errors (MSEs) corresponding to the MLE, LSE, WLSE, PCE, and MPSE methods.

Table3.1 The MSE and BIAS for different estimates of the GL distribution

α=1.5, β=1.2, λ=0.3, θ=0.2

n=100	Parameter	Mean	Bias	MSE
MLE	α	0.9048329	0.3967781	7.2621336
	B	4.336206	2.613505	6.116077
	λ	0.49415072	0.64716907	0.04361055
	θ	0.4868988	3.4344942	0.4772720
MPS	α	0.1473531	0.9017646	1.8552169
	B	2.0738082	0.7281735	1.1535780
	λ	0.0001087933	0.9996373558	0.0899350659
	θ	1.274000	5.370001	1.383114
LSE	α	1.0962723	0.2691518	1.3455058
	B	1.3805624	0.1504687	0.4904623
	λ	0.003123116	0.989589614	0.088506426
	θ	2.403890	11.019449	5.409376
WLSE	α	1.2517817	0.1654789	1.9637274
	B	1.4824150	0.2353458	0.6171359
	λ	0.001001431	0.996661895	0.089412644
	θ	2.520143	11.600717	1.981707
PE	α	1.551525671	0.034350448	0.004190934
	B	1.238487539	0.032072949	0.002657689
	λ	0.29056010	0.03146635	0.00152408
	θ	0.2034714252	0.0173571260	0.0001989665

Table3.2 The MSE and BIAS for different estimates of the GL distribution
α=1.5, β=1.2, λ=0.3, θ=0.2

n=75	Parameter	Mean	Bias	MSE
MLE	α	0.9709001	0.3527333	8.7927725
	B	3.830793	2.192328	6.303468
	λ	0.49628186 -	0.65427288	0.04745964
	θ	0.4863422	3.4317111	0.4766724
MPS	α	0.1674241	0.8883839	1.8146348
	B	2.093453	0.744544	1.219931
	λ	0.0001354118	0.9995486273	0.0899192689
	θ	1.288680	5.443399	1.433535
LSE	α	1.1347404	0.2435064	1.4986024
	B	1.4322266	0.1935222	0.5597740
	λ	0.003412733	0.988624223	0.088304599
	θ	2.337498	10.687488	1.185704
WLSE	α	1.1444146	0.2370569	1.8865839
	B	1.4262741	0.1885617	0.5540414
	λ	0.0009966123	0.9966779590	0.0894781318
	θ	2.548278	11.741391	2.089436
PE	α	1.55424799	0.03616533	0.00471765
	B	1.238225208	0.031854340	0.002878684
	λ	0.289248254	0.035839155	0.002009099
	θ	0.2051376738	0.0256883688	0.0002884604

Table3.3 The MSE and BIAS for different estimates of the GL distribution
α=1.5, β=1.2, λ=0.3, θ=0.2

n=50	Parameter	Mean	Bias	MSE
MLE	α	0.9778264	0.3481157	5.5407028
	B	3.216502	1.680418	4.404130
	λ	0.50319620 -	0.67732067	0.05497662
	θ	0.4870264	3.4351318	0.4781071
MPS	α	0.2208511	0.8527660	1.7247521
	B	2.076662	0.730552	1.262820
	λ	0.0002125432	0.9992915225	0.0898733016
	θ	1.355882	5.779410	1.649970
LSE	α	1.2198851	0.1867432	1.8962732
	B	1.5337218	0.2781015	0.7195194
	λ	0.005376949	0.982076835	0.091769266
	θ	2.229499	10.147494	2.768008
WLSE	α	1.1297127	0.2468582	1.2648045
	B	1.3920324	0.1600270	0.4308027
	λ	0.001036686	0.996544381	0.089389142
	θ	2.548069	11.740345	1.058470
PE	α	1.557257950	0.038171967	0.005396884
	B	1.237277906	0.031064922	0.003245306
	λ	0.289009581	0.036634731	0.002939778
	θ	0.2066572880	0.0332864401	0.0004429331

Table3.4 The MSE and BIAS for different estimates of the GL distribution
α=1.5, β=1.2, λ=0.3, θ=0.2

n=25	Parameter	Mean	Bias	MSE
MLE	α	0.7611466	0.4925689	8.8442867
	Β	2.539285	1.116070	12.279748
	λ	0.51783939 -	0.72613131	0.07631456
	θ	0.4880631	3.4403156	0.4795336
MPS	α	0.3491776	0.7672149	1.5677225
	Β	2.0309570	0.6924642	1.2729859
	λ	0.0003267596	0.9989108013	0.0898055936
	θ	1.535973	6.679863	2.283739
LSE	α	1.42036593	0.05308938	3.41176577
	Β	1.6833111	0.4027593	0.8869212
	λ	0.004005159	0.986649472	0.087844672
	θ	2.034122	9.170612	4.053129
WLSE	α	1.1899369	0.2067087	1.9332599
	Β	1.3993227	0.1661022	0.4150185
	λ	0.002960252	0.990132493	0.091366425
	θ	2.460887	11.304436	5.639558
PE	α	1.558890621	0.039260414	0.006283321
	Β	1.236393588	0.030327990	0.004365195
	λ	0.28814444	0.03951852	0.00615215
	θ	0.2109278203	0.0546391015	0.0009448704

Table4.1 The MSE and BIAS for different estimates of the GL distribution
α=1.5, β=1.2, λ=0.3, θ=0.2

n=100	**Parameter**	**Mean**	**Bias**	**MSE**
MLE	**α**	0.3043107	0.7971262	1.4690843
	Β	0.3014367	0.8325352	2.6480390
	λ	2.581409	4.162819	7.262855
	θ	2.470934	2.529906	7.583671
MPS	**α**	0.3237273	0.7841818	1.7235086
	Β	1.87671418	0.04261899	1.72810981
	λ	1.055910	1.111819	1.409608
	θ	0.2658383	1.3797690	1.8678291
LSE	**α**	1.0000072	0.3333285	2.9175419
	Β	2.1197488	0.1776382	1.6718920
	λ	0.3787584	0.2424832	0.7218128
	θ	0.0000590532	0.0000370176	0.0000864711
WLSE	**α**	0.2955218	0.8029854	1.4530903
	Β	1.85036119	0.02797844	1.59569768
	λ	0.07955152	0.84089696	0.18119697
	θ	1.281288	0.830412	1.252268
PE	**α**	1.565775303	0.043850202	0.007868576
	Β	1.82156775	0.01198208	0.00481100
	λ	0.495630745	0.008738510	0.002558163
	θ	0.72673403	0.03819147	0.01425817

Table4.2 The MSE and BIAS for different estimates of the GL distribution α=1.5, β=1.2, λ=0.3, θ=0.2

n=75	Parameter	Mean	Bias	MSE
MLE	α	0.3031669	0.7978888	1.7154938
	B	0.3584865	0.8008408	2.7092564
	λ	2.552353	4.104705	2.860143
	θ	2.329276	2.327537	١.384744
MPS	α	0.3221950	0.7852033	1.7306697
	B	1.89144645	0.05080358	1.58421271
	λ	1.047385	1.094770	1.301274
	θ	0.2559387	1.3656267	1.8108020
LSE	α	1.0282334	0.3145111	2.3876083
	B	2.069523	0.149735	1.451288
	λ	0.4075042	0.1849916	0.8426504
	θ	0.000191246	0.000701879	0.0000866158
WLSE	α	0.2903395	0.8064403	1.4660290
	B	1.84306189	0.02392327	1.53464258
	λ	0.08435725 -	0.83128550	0.17990432
	θ	0.1597025	1.2281464	1.787581242
PE	α	1.563590165	0.042393443	0.008781136
	B	1.82087558	0.01159755	0.00747620
	λ	0.495735910	0.008528180	0.003665816
	θ	0.73883703	0.05548146	0.02226933

Table4.3 The MSE and BIAS for different estimates of the GL distribution α=1.5, β=1.2, λ=0.3, θ=0.2

n=50	Parameter	Mean	Bias	MSE
MLE	α	0.2872034	0.8085310	1.5204198
	B	0.4419114	0.7544937	3.0235142
	λ	2.568863	4.137727	6.914156
	θ	2.395676	2.422394	8.859573
MPS	α	0.3927453	0.7381698	2.5370244
	B	1.96109657	0.08949809	2.21593804
	λ	1.085779	1.171558	1.657061
	θ	0.2416362	1.3451945	2.3755326
LSE	α	1.2092667	0.1938222	2.0703092
	B	2.056709	0.142616	3.324511
	λ	0.46245428	0.07509145	1.06962100
	θ	0.000823438	0.000605011	0.0000588792
WLSE	α	0.3050871	0.7966086	1.7274631
	B	1.82897914	0.01609952	1.81483316
	λ	0.09733546	0.80532908	0.18548993
	θ	25.15040	36.02914	5.8630710814
PE	α	1.56704181	0.04469454	0.01174677
	B	1.82163880	0.01202156	0.01837599
	λ	0.491612818	0.016774365	0.005761972
	θ	0.75784475	0.08263536	0.04294881

Table4.4 The MSE and BIAS for different estimates of the GL distribution α=1.5, β=1.2, λ=0.3, θ=0.2

n=25	Parameter	Mean	Bias	MSE
MLE	**α**	0.2596276	0.8269149	1.5509915
	B	0.6175583	0.6569121	3.5607627
	λ	2.543815	4.087629	6.808041
	θ	2.181663	2.116661	8.736413
MPS	**α**	0.6209367	0.5860422	5.0025167
	B	2.1667894	0.2037719	4.9645529
	λ	1.125755 -	1.251510	2.016021
	θ	0.1880552	1.2686503	3.1173965
LSE	**α**	1.3414075	0.1057283	3.7834733
	B	2.0856171	0.1586762	٢.9491522
	λ	0.45550939	0.08898122	1.01909303
	θ	0.000952456e	0.000217894	0.000287250
WLSE	**α**	0.4708161	0.6861226	3.5525853
	B	1.85348817	0.02971565	2.83778120
	λ	0.1579891	0.6840217	0.3273232
	θ	0.00235800	0.006052142	0.0000703276
PE	**α**	1.57401979	0.04934653	0.02298584
	B	1.82037790	0.01132106	0.02751918
	λ	0.48744084	0.02511832	0.01092097
	θ	0.8088111	0.1554444	0.1264593

From last Tables, The MSEs decrease as sample size increases.

Based on the comparative analysis of estimation techniques, the findings indicate that the PE method yields the most accurate estimates for the parameter α, the MPS approach performs best for estimating λ, and the MLE method shows superior performance for estimating θ, particularly when evaluated using mean squared error (MSE) across the majority of scenarios examined.

When ranked by mean squared error (MSE), the estimators show varying levels of effectiveness across parameters. For estimating α, the performance hierarchy is: PE, WLSE, MPS, LSE, and MLE. In the case of λ, the order shifts to: MPS, PE, WLSE, LSE, and MLE. For θ, the most accurate results are achieved by MLE, followed by PE, MPS, WLSE, and LSE.

6-Censored samples and types of censoring

In reliability testing, items selected from a population are subjected to experiments where their failure times are recorded. These tests often conclude either after a predetermined duration or once a specific number of failures occur, resulting in censored data.

Complete lifetime observations—referred to as uncensored data—are uncommon in practice, even under controlled conditions. Tests are frequently halted before all failures are observed due to constraints such as time or budget limitations.

An observation is considered right-censored at a value L when the exact failure time is unknown, but it is known to exceed or equal L. Conversely, left-censoring occurs at a value M when the failure time is only known to be less than or equal to M. Right-censoring is prevalent in survival and reliability studies, whereas left-censoring is relatively rare.

Common censoring mechanisms include:

- Type I censoring: Test ends at a fixed time.
- Type II censoring: Test stops after a set number of failures.

a) Type I censored data

A life test for a fixed number n of items is terminated when all the items have failed or at a reassigned timeT_0, whichever is the sooner. In this case the number of failures r is a random variable and T_0 is fixed.

the likelihood function can be written as

$$L(\Omega) = [\prod_{i=1}^{n} g(x;\Omega)^{I_i}]S(x;\Omega)^{n-\delta}$$

and the log-likelihood function is given by:

$$ln = n\,log\,\lambda + \sum\left[log\left(\frac{\alpha}{\beta}\right) - (\alpha+1)\,log\left(1+\frac{T}{\beta}\right)\right] + \alpha(\theta+1)\log\left(1+\frac{T}{\beta}\right)\left(\frac{\lambda}{\theta}\right) + \log\sum\left\{1-\left(1+\frac{T}{\beta}\right)^{\theta\alpha}\right\} + \left(\frac{\lambda}{\theta}\right)(n-k)\sum log\left\{1-\left(1+\frac{T_k}{\beta}\right)^{\theta\alpha}\right\}$$

$$\frac{\partial ln}{\partial\lambda} = \frac{n}{\lambda} + \frac{1}{\theta}\Sigma\,log\left\{1-\left(1+\frac{T}{\beta}\right)^{\alpha\theta}\right\} + \frac{n-k}{\theta}\Sigma\,log\left\{1-\left(1+\frac{T_k}{\beta}\right)^{\alpha\theta}\right\}$$

$$\frac{\partial ln}{\partial \lambda}=\frac{n}{\lambda}+\frac{1}{\theta}\Sigma\, log\left\{1-\left(1+\frac{T}{\beta}\right)^{\alpha\theta}\right\}+\frac{n-k}{\theta}\Sigma\, log\left\{1-\left(1+\frac{T_k}{\beta}\right)^{\alpha\theta}\right\}$$

$$\frac{\partial ln}{\partial \theta}=\Sigma\, \alpha log\left(1+\frac{T}{\beta}\right)\ -\frac{\lambda}{\theta}\Sigma\frac{\alpha\left(1+\frac{T}{\beta}\right)^{\alpha\theta} log\left(1+\frac{T}{\beta}\right)}{1-\left(1+\frac{T}{\beta}\right)^{\alpha\theta}}-\frac{\lambda}{\theta^2}\Sigma\log\left\{1-\left(1+\frac{T}{\beta}\right)^{\alpha\theta}\right\}$$
$$+\frac{\lambda(n-k)}{\theta^2}\Sigma\, log\left\{1-\left(1+\frac{T_k}{\beta}\right)^{\alpha\theta}\right\}$$

$$\frac{\partial ln}{\partial \beta}=-\frac{n}{\beta}+\frac{(\alpha+1)T}{\beta^2+\beta T}-\frac{(\alpha+1)T}{\beta^2+\beta T}-\frac{\lambda\alpha\left(1+\frac{T}{\beta}\right)^{\alpha\theta-1} t}{\beta^2\left[1-\left(1+\frac{T}{\beta}\right)^{\alpha\theta}\right]}+\frac{\lambda(n-k)\alpha\left(1+\frac{T_k}{\beta}\right)^{\alpha\theta-1} T_k}{\beta^2\left\{1-\left(1+\frac{T_k}{\beta}\right)^{\alpha\theta}\right\}}$$

$$\frac{\partial ln}{\partial \alpha}=\frac{n}{\alpha}-log\left(1+\frac{T}{\beta}\right)\ +(\alpha+1)\log\left(1+\frac{T}{\beta}\right)\ -\lambda\frac{\left(1+\frac{T}{\beta}\right)^{\alpha\theta} log\left(1+\frac{T}{\beta}\right)}{1-\left(1+\frac{T}{\beta}\right)^{\alpha\theta}}$$
$$-\frac{\lambda(n-k)\left(1+\frac{T}{\beta}\right)^{\alpha\theta} log\left(1+\frac{T_k}{\beta}\right)}{\left\{1-\left(1+\frac{T_k}{\beta}\right)^{\alpha\theta}\right\}}$$

b) Type I I censored data

A life test on n items is terminated after a predetermined number of failures r have occurred (r < n). Here the observations will present themselves as an ordered sequence, the shortest life time being observed first and the r-th shortest being observed last. In this case r and n are fixed beforehand and $X_{(r)}$, the time of the r-th failure, is a random variable. The likelihood function then becomes

$$L(\Omega)=\frac{n!}{t!}[\prod_{i=1}^{n-t} g(x_i;\Omega)][S(x_{n-t};\Omega)]^t$$

where x_i is the order statistic of order i, and the log-likelihood function, expressed in terms of the original baseline distribution, reads

$$ln = n\, log\, \lambda + \Sigma\left[log\left(\frac{\alpha}{\beta}\right) - (\alpha+1)\, log\left(1+\frac{t}{\beta}\right)\right] + \alpha(\theta+1)\log\left(1+\frac{t}{\beta}\right)\left(\frac{\lambda}{\theta}\right) + \log\sum\left\{1-\left(1+\frac{t}{\beta}\right)^{\theta\alpha}\right\} + \left(\frac{\lambda}{\theta}\right)(n - k)\sum log\left\{1-\left(1+\frac{t_k}{\beta}\right)^{\theta\alpha}\right\}$$

$$\frac{\partial ln}{\partial\lambda} = \frac{n}{\lambda} + \frac{1}{\theta}\Sigma\, log\left\{1-\left(1+\frac{t}{\beta}\right)^{\alpha\theta}\right\} + \frac{n-k}{\theta}\Sigma\, log\left\{1-\left(1+\frac{t_k}{\beta}\right)^{\alpha\theta}\right\}$$

$$\frac{\partial ln}{\partial\theta} = \Sigma\, \alpha log\left(1+\frac{t}{\beta}\right) - \frac{\lambda}{\theta}\Sigma\frac{\alpha\left(1+\frac{t}{\beta}\right)^{\alpha\theta} log\left(1+\frac{t}{\beta}\right)}{1-\left(1+\frac{t}{\beta}\right)^{\alpha\theta}} - \frac{\lambda}{\theta^2}\Sigma\log\left\{1-\left(1+\frac{t}{\beta}\right)^{\alpha\theta}\right\} + \frac{\lambda(n-k)}{\theta^2}\Sigma\, log\left\{1-\left(1+\frac{t_k}{\beta}\right)^{\alpha\theta}\right\}$$

$$\frac{\partial ln}{\partial\beta} = -\frac{n}{\beta} + \frac{(\alpha+1)t}{\beta^2+\beta t} - \frac{(\alpha+1)t}{\beta^2+\beta t} - \frac{\lambda\alpha\left(1+\frac{t}{\beta}\right)^{\alpha\theta-1} t}{\beta^2\left[1-\left(1+\frac{t}{\beta}\right)^{\alpha\theta}\right]} + \frac{\lambda(n-k)\alpha\left(1+\frac{t_k}{\beta}\right)^{\alpha\theta-1} t_k}{\beta^2\left\{1-\left(1+\frac{t_k}{\beta}\right)^{\alpha\theta}\right\}}$$

$$\frac{\partial ln}{\partial\alpha} = \frac{n}{\alpha} - log\left(1+\frac{t}{\beta}\right) + (\alpha+1)\log\left(1+\frac{t}{\beta}\right) - \lambda\frac{\left(1+\frac{t}{\beta}\right)^{\alpha\theta} log\left(1+\frac{t}{\beta}\right)}{1-\left(1+\frac{t}{\beta}\right)^{\alpha\theta}} - \frac{\lambda(n-k)\left(1+\frac{t}{\beta}\right)^{\alpha\theta} log\left(1+\frac{t_k}{\beta}\right)}{\left\{1-\left(1+\frac{t_k}{\beta}\right)^{\alpha\theta}\right\}}$$

Table.5.1 MLE Estimation under type-I censoring scheme Time of censoring (30 %) α=2.5, β=2, λ=0.5, θ=1.2

n=25	Par	Bias	MSE	CI-Lower	CI-Upper	CP
n=25	**α**	0.07434857	0.02521572	2.03943042	2.25796387	98.9
	Β	0.05525631	0.01237550	2.08551849	2.13550673	99.0
	λ	0.00962857	0.00481463	0.00666755	0.00047529	99.6
	θ	0.3155546	0.1042908	1.1808884	1.4502207	99.6
n=50	**α**	0.07302460	0.02577747	2.01528048	2.27681791	99.0
	Β	0.05444400	0.01204797	2.08176051	2.13601548	98.7
	λ	0.0963375	0.0481721	0.00650901	0.00011638	99.3
	θ	0.29721769	0.09256805	1.16968543	1.42474996	99.3
n=75	**α**	0.07245479	0.02735774	1.98853786	2.30128128	98.60
	Β	0.05412594	0.01199000	2.07593899	2.14056477	98.90
	λ	0.996358334	0.248182493	0.001619205	0.002022461	99.30
	θ	0.28981059	0.08927443	1.14726390	1.43235728	99.30
n=100	**α**	0.07061974	0.02441367	2.01020674	2.27227222	98.50
	Β	0.05439700	0.01202688	2.08171151	2.13587648	98.10
	λ	0.00963567	0.00481817	0.000637031	0.000006284	99.2
	θ	0.29038029	0.08874869	1.15989302	1.42086757	99.20

Table.5.2 MLE Estimation under type-I censoring scheme Time of censoring (60 %) α=2.5, β=2, λ=0.5, θ=1.2

n=25	Par	Bias	MSE	CI-Lower	CI-Upper	CP
n=25	α	0.07807586	0.02440765	2.14648672	2.16581672	98.00
	B	0.05474342	0.01201894	2.09846819	2.12050549	98.90
	λ	0.8862666	0.00481368	0.00748323	0.00985112	9.86
	θ	0.3206271	0.1046146	1.2371341	1.4041201	98.60
n=50	α	0.07927127	0.02515799	2.14929374	2.16779136	99.70
	B	0.05337433	0.01142459	2.09613117	2.11736616	99.50
	λ	0.0963252	0.0481660	0.00723876	0.00950945	99.40
	θ	0.29995745	0.09164148	1.21989404	1.38002086	99.40
n=75	α	0.08012036	0.02569503	2.15193420	2.16854725	99.80
	B	0.05240866	0.01101059	2.09522737	2.11440727	96.70
	λ	0.0963667	0.0481866	0.00714422	0.00918889	96.30
	θ	0.28531938	0.08275866	1.21322930	1.35740945	96.30
n=100	α	0.08048630	0.02592894	2.15294423	2.16900096	99.70
	B	0.05198073	0.01083032	2.09469361	2.11322932	94.10
	λ	0.00963848	0.0481957	0.00708751	0.00906428	93.90
	θ	0.27892473	0.07906214	1.20923169	1.34861777	93.90

Table.5.3 MLE Estimation under type-I censoring scheme Time of censoring (90 %) α=2.5, β=2, λ=0.5, θ=1.2

n=25	Par	Bias	MSE	CI-Lower	CI-Upper	CP
n=25	α	0.1606836	0.3875044	1.2759288	3.3668055	97.70
	B	0.04042945	0.10982463	1.45064634	2.71107145	99.20
	λ	0.990922422	0.246718350	0.064416748	0.073494326	99.40
	θ	0.4303300	1.3845480	-0.7172092	3.5778691	97.99
n=50	α	0.1297063	0.1017831	1.8952455	2.6235797	99.90
	B	0.04867311	0.01114762	2.01717903	2.17751341	99.90
	λ	0.995735256	0.247946438	0.014766253	0.019030997	99.90
	θ	0.3225270	0.1868147	0.7582959	1.8867581	99.90
n=75	α	0.12058231	0.07005224	2.02732398	2.45500527	100.0
	B	0.00948807	3.00989098	2.071747	2.126206	100.0
	λ	0.0962781	0.0481425	0.00790032	0.00931885	99.0
	θ	0.31437645	0.09947656	1.26461312	1.36413978	99.0
n=100	α	0.11987336	0.06931358	2.02641713	2.45307632	100.0
	B	0.04964816	0.01004502	2.07260589	2.12598677	100.0
	λ	0.0962762	0.0481416	0.00796077	0.00927694	98.7
	θ	0.31503233	0.09979986	1.26885679	1.36120786	98.7

Table.5.4 MLE Estimation under type-I censoring scheme Time of censoring (99.9 %)
α=2.5, β=2, λ=0.5, θ=1.2

n=25	Par	Bias	MSE	CI-Lower	CI-Upper	CP
n=25	**α**	0.4371653	3.3743646	-0.2936183	6.0422795	97.00
	B	0.2641068	9.5834347	-3.4532927	8.5097197	95.36
	λ	0.990052121	0.245776465	-0.047850295	0.057798174	99.50
	θ	1.106916	7.915133	-2.965042	7.178875	93.65
n=50	**α**	0.3833146	2.1224647	0.3373167	5.1959416	98.60
	B	0.03143436	4.21547154	1.96137925	6.08711667	98.10
	λ	0.993484030	0.247061167	0.031186475	0.037702445	99.70
	θ	0.6615023	2.9177210	-1.4266841	4.7496887	97.29
n=75	**α**	0.3032391	1.1647422	0.8559287	4.3570279	97.6
	B	0.01888343	0.69911872	0.32429560 -	3.60017068	99.5
	λ	0.9959549800	0.2479837699	0.0008789953	0.0049240153	97.6
	θ	0.4129593	1.0016378	-0.3747325	3.2006510	97.4
n=100	**α**	0.2914884	1.2648131	0.6970506	4.4689032	99.0
	B	0.006124294	0.465132327	0.650590855	3.324911968	99.0
	λ	0.9961867805	0.2480984102	0.0004009682	0.0042141877	95.9
	θ	0.33654066	0.63446782	-0.07915848	2.75223980	90.

7. Conclusion

The Gompertz-Lomax distribution is a powerful and flexible model that extends the capabilities of its parent distributions. Its ability to model various hazard shapes and accommodate heavy tails makes it a valuable tool in statistical modeling. With robust estimation techniques and growing literature, it continues to find applications in diverse fields.

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