# Estimation and Prediction of Logistic Nadarajah-Haghighi distribution

***By***

**Dr. Mohamed Mohamed Abdelkader**

Lecturer of Statistics, Department of Statistics, Al-Azhar University

mkader77@azhar.edu.eg

**Dr. Tamer Hassan Mohamed**

Lecturer of Statistics, Department of Statistics, Al-Azhar University

Tamerhassan78@hotmail.com

***Scientific Journal for Financial and Commercial Studies and Research (SJFCSR)***

***Faculty*** of Commerce – Damietta University

Vol.5, No.2, Part 1., July 2024

**APA Citation:**

**Abdelkader**, M. M. and **Mohamed**, T. H. (2024). Estimation and Prediction of Logistic Nadarajah-Haghighi distribution, *Scientific Journal for Financial and Commercial Studies and Research*, Faculty of Commerce, Damietta University, 5(2)1, 553-575.

https://cfdj.journals.ekb.eg/ **Website:**

# Estimation and Prediction of Logistic Nadarajah-Haghighi distribution

***Dr. Mohamed Mohamed Abdelkader and Dr.Tamer Hassan Mohamed***

## Summary

This study explores four estimation methods for the Logistic Nadarajah-Haghighi (LNH) distribution, utilizing complete sampling and Maximum Likelihood Estimation (MLE) based on two censoring types: | & ||. The study assesses the squared bias and variances of these estimates through Monte Carlo simulations. Additionally, it provides insights into ordinary moments, the quintile function, mean residual life, and Renyi entropy.

**Keywords:** Logistic Nadarajah-Haghighi - Renyi entropy - mean residual life - Progressive Type-II Censoring

## 1.Introduction

The Nadarajah-Haghighi (NH) distribution, proposed as a generalized form of the exponential distribution by Nadarajah and Haghighi (2011) [18], offers an alternative to conventional distributions like gamma, Weibull, and exponentiated exponential. The distribution has seen successful extensions for more accurate statistical modelling and inference.

The parent Nadarajah-Haghighi distribution (with parameters $\alpha, \beta > 0$) has pdf given by:

$$g(x;\alpha,\beta) = \alpha\beta(1+\beta x)^{\alpha-1}e^{1-(1+\beta x)^{\alpha}}, \qquad (1)$$

and cdf given by:

$$G(x) = 1 - e^{1-(1+\beta x)^{\alpha}} \quad where.x > 0, \alpha, \beta > 0 \qquad (2)$$

The research builds upon previous work in this domain. Ziyad (2017) [21] focused on statistical inferences for the exponentiated Nadarajah-Haghighi (ENH) distribution with progressively type-II censoring. Cícero (2018a) [2] proposed the beta Nadarajah-Haghighi distribution as a generalization. Hilany (2018b) [4] derived ordinary differential equations (ODE) for the probability functions of the Marshall-Olkin Nadarajah-Haghighi distribution using differentiation. Morad (2018c) [15] introduced the Extended Exponentiated-Nadarajah-Haghighi (EENH) distribution as a four-parameter model. Hisham (2019a) [5] proposed the Burr X Nadarajah-Haghighi distribution, expanding the original distribution. Ms.Sana (2019b) [16] delved into maximum likelihood and Bayesian estimation of the two unknown parameters of the Nadarajah and Haghighi distribution. Fernando (2020a) [3] presented the logistic Nadarajah–Haghighi distribution, a three-parameter model, and proposed a parametric regression model. Marija (2020b) [12] estimated shape and scale parameters of Nadarajah-Haghighi based on simple random samples (SRS) and ranked set sampling (RSS). Sanku (2021) [20] presented estimation methods for parameters and acceleration factor in Nadarajah–Haghighi distribution based on constant-stress partially accelerated life tests. Almarashi (2022a) [1] focused on maximum likelihood and Bayesian estimation methods for the unknown parameters, reliability, and hazard rate functions of the Nadarajah–Haghighi (NH) distribution, employing adaptive type-I progressive hybrid censoring (ATIPHC) scheme. Muhammad (2022b) [17] proposed and studied a generalization of the inverted Nadarajah–Haghighi distribution.

## 2.Logistic-X Family

M. A. Aljarrah. (2014) [13] introduced a comprehensive approach for constructing distribution families. Their method involves utilizing a probability density function (PDF) denoted as r(.) for a continuous random variable. By applying a function $W(\mathrm{G}(x))$, which adheres to specified conditions, they systematically create the T-X family. The cumulative distribution function (CDF) of this T-X family is explicitly defined as part of their innovative methodology as follows:

$$\mathrm{F(x)} = \int_{a}^{w[\mathrm{G}(x)]} \mathrm{r(t)}dt \ , \qquad (3)$$

where

$W\left[G\left(x\right)\right]$ is differentiable and monotonically non-decreasing, and $W\left[G\left(x\right)\right] \to a$ as $x \to -\infty$ *and* $W\left[G\left(x\right)\right] \to b$ *as* $x \to \infty$. The pdf corresponding to (3) is given by:

$$f(x) = \{\tfrac{d}{dx} w[G(x)]\} r\{w[G(x)]\} \qquad (4)$$

M. H. Tahir (2016) [11] introduced the Logistic-X Family, a novel family of distributions. This family is characterized by T, a random variable that follows a logistic distribution with a specified shape parameter.

Let $G(x;\varphi)$ be the baseline cdf on a parameter vector $\varphi$. we define the cdf of the LX family by

$$\mathrm{F}(\mathrm{x};\lambda,\varphi)=[1+\{-\log[\bar{G}(x;\varphi)]\}^{-\lambda}]^{-1}. \quad (5)$$

The Lx family pdf is expressed as

$$\mathrm{f}(\mathrm{x};\lambda,\varphi)=\frac{\lambda g(x;\varphi)}{\bar{G}(x;\varphi)}\{-\log[\bar{G}(x;\varphi)]\}^{-\lambda-1}[1+\{-\log[\bar{G}(x;\varphi)]\}^{-\lambda}]^{-2}. \quad (6)$$

## 3.Logistic Nadarajah-Haghighi distribution (LNH).

By replacing (1) and (2) in (5) and (6) the cdf and pdf of the Logistic Nadarajah-Haghighi distribution **(LNH**) distribution are given by:

$$F(x;\alpha,\beta,\lambda)=[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{-1}, \quad for\ x>0;\alpha,\beta,\lambda>0 \quad (7)$$

and

$$f(x;\alpha,\beta,\lambda)=\alpha\beta\lambda(1+\beta x)^{\alpha-1}[(1+\beta x)^{\alpha}-1]^{-\lambda-1}[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{-2}. \quad (8)$$

The survival function and the hazard function of the LNH distribution are given by

$$s(x)=1-[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{-1}, \quad (9)$$

and

$$h(x)=\frac{\alpha\beta\lambda(1+\beta x)^{\alpha-1}[(1+\beta x)^{\alpha}-1]^{-\lambda-1}[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{-2}}{1-[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{-1}}. \quad (10)$$

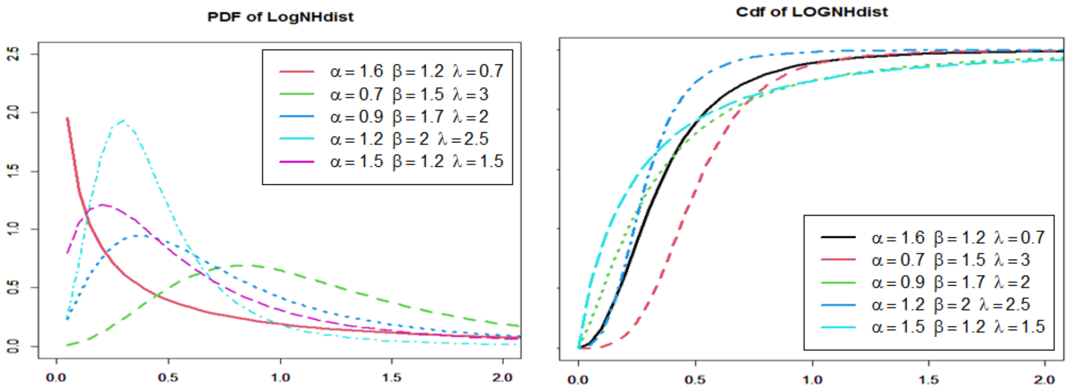


**Fig. 1:** Plots of the pdf $f(x)$ and cdf $F(x)$ of the LNH distribution.

Based on Figure 1, we can observe that the LNH distribution displays three distinct shapes: decreasing, bathtub, and unimodal. Furthermore, the newly proposed model exhibits similarities to other well-known distributions, such as Weibull, Gamma, and Exponential. These resemblances emerge when specific parameter values are chosen.

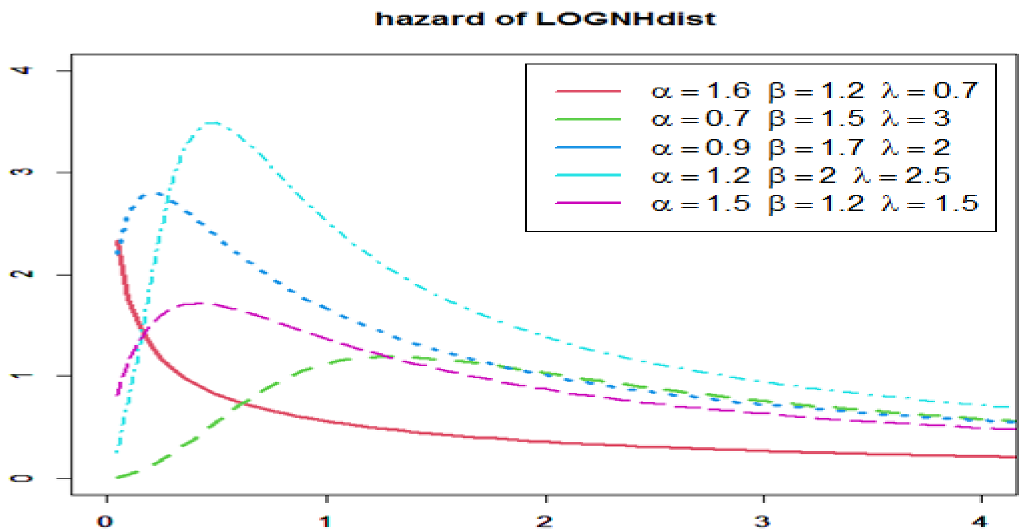


**Fig. 2**: Hazard Rate Function Plot of the LNH distribution.

## 4.Quantile function

The quantile function, denoted as Q(p) or Q for the logistic Nadarajah-Haghighi distribution, is determined by solving the equation G(Q) = P, where G represents the cumulative distribution function. The quantile function Q, when evaluated at a vector p of percentiles, provides the corresponding values that satisfy this equation:

$$Q = \frac{[1 + (\frac{1}{p} - 1)^{\frac{-1}{\lambda}}]^{\frac{1}{\alpha}}}{\beta} - \frac{1}{\beta} \qquad \alpha, \beta, \lambda > 0 \qquad (11)$$

The r th moment of the logistic Nadarajah-Haghighi distribution can be derived from equation (11) by replacing Q by the variable x and replacing p by u. Here, p represents a uniform random variable in the range (0,1) as follows:

$$x_u = \frac{[1 + (\frac{1}{u} - 1)^{\frac{-1}{\lambda}}]^{\frac{1}{\alpha}}}{\beta} - \frac{1}{\beta} \qquad \alpha, \beta, \lambda > 0 \qquad (12)$$

The Bowley Skewness measure (Bsk) and the Moor's Kurtosis measure (Mkur) [14] are defined by:

$$Bsk = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}},$$

$$Mkur = \frac{Q_{0.875} - 2Q_{0.625} - 2Q_{0.375} + Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

Table 1 presents the skewness and kurtosis values for the logistic Nadarajah-Haghighi distribution under different parameter settings.

**Table.1 Quantile for different parameter value.**

| **α, β, λ** | $Q_{0.125}$ | $Q_{0.25}$ | $Q_{0.325}$ | $Q_{0.5}$ | $Q_{0.625}$ | $Q_{0.75}$ | $Q_{0.875}$ | **skewn** | **kurtosis** |
|---|---|---|---|---|---|---|---|---|---|
| **2.5,3,2.4** | 0.05283 | 0.07221 | 0.08913 | 0.10650 | 0.12667 | 0.15370 | 0.20076 | 0.15845 | 0.46373 |
| **2,1.5,2.3** | 0.13030 | 0.18192 | 0.22797 | 0.27614 | 0.33304 | 0.41084 | 0.54996 | 0.17682 | 0.52094 |
| **2.5,3,2** | 0.04561 | 0.06666 | 0.08596 | 0.10650 | 0.13106 | 0.16495 | 0.22590 | 0.18923 | 0.55435 |
| **0.9,1.7,1.4** | 0.16491 | 0.30491 | 0.46853 | 0.68242 | 0.99684 | 1.54771 | 2.94027 | 0.39248 | 1.31944 |
| **2.5,1.7,0.9** | 0.02620 | 0.06409 | 0.11576 | 0.18795 | 0.29518 | 0.47472 | 0.87082 | 0.39675 | 1.18373 |
| **1.2,1.5,0.9** | 0.06335 | 0.16028 | 0.30260 | 0.52120 | 0.88879 | 1.62027 | 3.75747 | 0.50558 | 1.80099 |
| **0.5,1,0.7** | 0.12794 | 0.45965 | 1.19641 | 3.00000 | 8.45294 | 32.68626 | 291.99005 | 0.84234 | 8.76507 |

## 5.Raw moment

the 1-st raw moment of the Random variable X is [9]:

$\mu_{x=E(x)=\int xf(x)\,dx},$

and the r-th raw moment of the LNH Random variable X is

$$\acute{\mu_r} = E(x^r) = \alpha\beta\lambda \int_0^{\infty} x^r (1+\beta x)^{\alpha-1}[(1+\beta x)^{\alpha} - 1]^{-\lambda-1}[1 + \{(1+\beta x)^{\alpha} - 1\}^{-\lambda}]^{-2} dx. \quad (13)$$

We can calculate skewness and kurtoses from moments as:

$$sk = \frac{\mu_3^2}{\mu_2^3},$$

$$ku = \frac{\mu_4}{\mu_2^2}.$$

Table 2 presents the moment, mean, variance, skewness, and kurtosis values for the logistic Nadarajah-Haghighi distribution under different parameter settings.

**Table.2 moment, mean, variance, skewness and kurtosis**

| **α, β, λ** | **0.6,1.3,2.8** | **1.4,1.1,0.9** | **0.8,1.6,2.1** | **1,1.8,2.3** |
|---|---|---|---|---|
| **Mean** ($\mu$) | 2.053321 | 1.151684 | 1.228858 | 0.750808 |
| **m2** ($\mu_2'$) | 6.814764 | 4.206815 | 3.126969 | 1.122327 |
| **m3** ($\mu_3'$) | 32.39307 | 23.68192 | 13.65019 | 3.405697 |
| **m4** ($\mu_4'$) | 195.4822 | 161.8371 | 81.328 | 16.96168 |
| **Variance** ($\mu_2$) | 2.598637 | 2.880439 | 1.616877 | 0.558615 |
| **CV** | 0.785084 | 1.473656 | 1.034753 | 0.995468 |
| **sk** | 1.844917 | 2.49605 | 2.837484 | 4.129757 |
| **kur** | 4.181049 | 6.755639 | 10.66426 | 27.68829 |

## 6.Mean residual life

Reliability analysts, statisticians, and other professionals have demonstrated increased interest in the concept of mean residual life (MRL) [10]. When considering an entity with an age of t, the remaining life beyond time t is a stochastic variable. The mean residual life at time t is corresponding to the expected value of this random residual life.

Let F be a life distribution (i.e., $F(t)=0$ for $t<0$) with a finite first moment. Let $\bar{F}(t)=1-F(t)$. $X$ is the random life with distribution F. The MRL function is defined as:

$$m(t)=E(x-t/x>t)$$

When $\bar{F}(t)>0$, We can express m(t) in the form:

$$m(t)=\int_0^\infty uf(u)du/\bar{F}(t)-\text{t}.$$

Like the failure rate function (recall that it is defined as $r(t) = f(t)/\bar{F}(t)$ when $\bar{F}(t) > 0$), the MRL function is a conditional concept. Both functions are conditioned on survival to time t.

The mean residual life at time t of the LNH Random variable X is

$$m(t) = \frac{1}{1-[1+\{(1+\beta t)^{\alpha}-1\}^{-\lambda}]^{-1}} \int_{t}^{\infty} 1-[1+\{(1+\beta t)^{\alpha}-1\}^{-\lambda}]^{-1} dt \text{..........(14)}$$

Table 3 presents the Empirical Mean Residual life values for the logistic Nadarajah-Haghighi distribution under different parameter settings.

**Table.3. Empirical Mean Residual life. $\alpha = 1.6, \beta = 1.2, \lambda = 0.7. n = 50$**

| No | Death. Time | EMRL | No | Death. Time | EMRL |
|---|---|---|---|---|---|
| **1** | 0.000681 | 2.195853 | **11** | 0.09163 | 2.654152 |
| **2** | 0.005547 | 2.236633 | **12** | 0.099936 | 2.715474 |
| **3** | 0.00678 | 2.282961 | **13** | 0.111459 | 2.777031 |
| **4** | 0.027444 | 2.311478 | **14** | 0.116634 | 2.848852 |
| **5** | 0.047444 | 2.342399 | **15** | 0.117984 | 2.928859 |
| **6** | 0.059083 | 2.383732 | **16** | 0.131646 | 3.000939 |
| **7** | 0.062266 | 2.435911 | **17** | 0.13477 | 3.088657 |
| **8** | 0.070895 | 2.485074 | **18** | 0.140341 | 3.179433 |
| **9** | 0.08215 | 2.534157 | **19** | 0.150173 | 3.271845 |
| **10** | 0.0914 | 2.588029 | **20** | 0.152099 | 3.378917 |

As observed, the Empirical Mean Residual Life value needs to significantly increase with sample size to enable powerful procedures.

## 7.Renyi Entropy

It is possible to show that Entropy measure [19] extends to case of continuous random variable as follows:

$$R_{\alpha}(x) = \lim_{n\to\infty}(\mathrm{I}_{\alpha}(\mathrm{p}_n) - \log n) = \frac{1}{1-\alpha}\log\int p^{\alpha}(x)dx \ ,$$

and the quadratic entropy takes the form:

$$R_2(x) = -\log\int p^2(x)dx \ .$$

Notice, however, that now entropy is no longer positive, in fact it can become arbitrarily large negative.

Entropy measure of the LNH Random variable X is

$$R_{\alpha}(x) = \frac{1}{1-\alpha}\log\int(\alpha\beta\lambda(1+\beta x)^{\alpha-1}[(1+\beta x)^{\alpha}-1]^{-\lambda-1}[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{-2})^{\alpha}dx \text{...(15)}$$

Table 4 presents the Renyi Entropy values for the logistic Nadarajah-Haghighi distribution under different parameter settings.

**Table. 4. Renyi Entropy**

| parameter | v = 1.5 | v = 2 | v = 3 |
|---|---|---|---|
| (0.9,1.7,2) | 0.954233 | 0.844134 | 0.72618 |
| (1.5,1.2,1.8) | 0.704143 | 0.573664 | 0.438247 |
| (1.2,2,2.5) | 0.33635 | 0.223088 | 0.101986 |
| (0.9,1.2,0.7) | 1.189355 | 0.520089 | -0.5375 |
| (1.6,1.2,0.7) | 0.408512 | -0.12698 | -1.11977 |
| (0.7,1.5,3) | -0.10747 | -0.20814 | -0.31834 |

It's important to recognize that as the Renyi entropy value increases, the level of uncertainty in the system also grows.

## 8.Methods of Estimation

This section involves the estimation of parameters for the Logistic Nadarajah-Haghighi (LNH) distribution using a complete sample technique. Five distinct methods are employed for this estimation: Maximum Likelihood (MLE), Least-Squares (LS), Weighted Least Squares, Maximum Product of Spacing (MPS), and Percentile-Based Estimation (PE). The effectiveness of each method is evaluated using the R software.

## 8.1-Maximum Likelihood Estimation

let $x_1,.......;x_n$ be a random sample of size n from X. Here, we obtain the MLEs of the unknown parameters of the LX family from complete samples only. The total log-likelihood function for Θ is given by

$$\ell(\Theta) = n\log\lambda + \sum_{i=1}^{n}\log[h_g(x;\varphi)] - (\lambda+1)\sum_{i=1}^{n}\log[H_g(x;\varphi)] - 2\sum_{i=1}^{n}\log(1+[H_g(x;\varphi)]^{-\lambda})$$

In LNH is given by :

$$\ell(\Theta)$$

$$\ell(\Theta) = n\,log\,\lambda + \sum_{i=1}^{n} log[\,\alpha\beta(1+\beta\,x)^{\alpha-1}] - (\lambda+1)\sum_{i=1}^{n} log[\,(1+\beta\,x)^{\alpha} - 1] - 2\sum_{i=1}^{n} log(\,1+[(1+\beta\,x)^{\alpha}-1]^{-\lambda}), \qquad (16)$$

For obtaining the partial derivatives, differentiating (18) with respect to $\alpha,\beta$ and $\lambda$, we get:

$$U_\lambda = \tfrac{n}{\lambda} - \sum_{i=1}^{n}\log[(1+\beta x)^{\alpha}-1] + 2\sum_{i=1}^{n}\log(1+[(1+\beta x)^{\alpha}-1]^{-\lambda})^{-1}\{\log[(1+\beta x)^{\alpha}-1]$$

,

$$U_\alpha=\sum_{i=1}^{n}[\frac{\beta(1+\beta x)^{\alpha-1}(\alpha\, log(1+\beta x)+1)}{\alpha\beta(1+\beta x)^{\alpha-1}}]-(\lambda+1)\sum_{i=1}^{n}[\frac{(1+\beta x)^{\alpha}\, log(1+\beta x)}{(1+\beta x)^{\alpha}-1}]+2\lambda\sum_{i=1}^{n}[\frac{(1+\beta x)^{\alpha}-1}{((1+\beta x)^{\alpha}-1)\{1+[(1+\beta x)^{\alpha}-1]^{\lambda}\}}],$$

and

$$U_\beta=\sum_{i=1}^{n}[\frac{\alpha(1+\beta x)^{\alpha}(1+\alpha\beta x))}{\alpha\beta(1+\beta x)^{\alpha-1}}]-(\lambda+1)\sum_{i=1}^{n}[\frac{\alpha\, x(1+\beta x)^{\alpha-1}}{(1+\beta x)^{\alpha}-1}]+2\lambda\sum_{i=1}^{n}[\frac{\alpha\, x(1+\beta x)^{\alpha-1}}{((1+\beta x)^{\alpha}-1)\{1+[(1+\beta x)^{\alpha}-1]^{\lambda}\}}].$$

## 8.2-Method of Ordinary Least Squares (L):

The best estimates according to LS method [6] are those which minimize the following quantity:

$$Q_1=\sum_{i=1}^{n}(G(x_{(i)}-\frac{i}{n+1}))^2,$$

with respect to α, β and λ. $x_{(i)}$ is the I-th orders statistic of LNH

Similarly, these estimators are also obtained by solving the following equation (for k = 1 ,2 ,3 ,4) (see [6]) as:

$$Q_1=\{[1+\{(1+\beta(x_i-\tfrac{i}{n+1}))^{\alpha}-1\}^{-\lambda}]^{-1}\}\psi_k(x_{(i)}/\Omega)=0$$

Where

$$\psi_1(x_{(i)}/\Omega)=\frac{\partial G(x)}{\partial\lambda}=-(w^{\alpha}-1)^{-\lambda}[1+(w^{\alpha}-1)^{-\lambda}]^{-2}\, log(w^{\alpha}-1),$$

$$\psi_2(x_{(i)}/\Omega)=\frac{\partial G(x)}{\partial\alpha}=\lambda w^{\alpha}\, log\, w\,(w^{\alpha}-1)^{-\lambda-1}[1+(w^{\alpha}-1)^{-\lambda}]^{-2},$$

$$\psi_3(x_{(i)}/\Omega)=\frac{\partial G(x)}{\partial\beta}=\alpha\lambda(x_i-\frac{i}{n+1})w^{\alpha-1}(w^{\alpha}-1)^{-\lambda-1}[1+(w^{\alpha}-1)^{-\lambda}]^{-2}.$$

where $w=1+\beta(x_i-\frac{i}{n+1})$ .The solution of $\psi_k$ for k = 1,2,3,4 may be obtained numerically.

## 8.3-Method of Weighted Least Squares (WLS):

The WLS estimators of α, β and λ. of LNH distribution can be obtained by minimizing the quantity:

$$Q_2=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{n-i+1}(G(x_{(i)}-\frac{i}{n+1}))^2$$

with respect to α., β, and λ In LNH $Q_2$ is given by

$$Q_2=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{n-i+1}([1+\{(1+\beta(x_i-\tfrac{i}{n+1}))^{\alpha}-1\}^{-\lambda}]^{-1})^2 .$$

## 8.4-Method of Percentile Estimation (PCE) :

This method has been introduced by kao [7, 8]. The PCEs estimators of α, β, and λ of LNH distribution can be obtained by minimizing the quantity:

$$Q_3=\sum_{i=1}^{n}[x_{(i)}-\frac{[1+(\frac{1}{p}-1)^{\frac{-1}{\lambda}}]^{\frac{1}{\alpha}}}{\beta}+\frac{1}{\beta}]^2$$

## 9.Simulation Study

To assess the performance of the proposed estimation methods, a simulation study is conducted through Monte Carlo analysis. This involves generating 5000 random data points from the Logistic Nadarajah-Haghighi (LNH) distribution under specific conditions:

1. Sample sizes are $n=50,100,150,300$
2. Assuming the following selected cases of parameters $\alpha, \beta$ *and* $\lambda$ of the LNH distribution:
    a. $\alpha=0.9, \beta=1.7, \lambda=2$
    b. $\alpha=0.7, \beta=1.5, \lambda=3$
    c. $\alpha=1.2, \beta=2, \lambda=2.5$
    d. $\alpha=1.5, \beta=1.2, \lambda=1.7$

Utilizing the generated data and employing diverse estimation methods, Tables (5) to (8) present the results of the Mean Square Error (MSE) and Relative Biases (BIAS) for each of the five estimation methods.

**Table.5 The MSE and BIAS for different estimates of the LNH distribution with different values**

| Estimation | parm | α = 0.7 | β = 1.5 | λ = 3 | parm | α = 0.7 | β = 1.5 | λ = 3 |
|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | | n = 150 | | | |
| MLE | Bias | 1.300111 | 2.32581 | 0.238748 | Bias | 0.474887 | 1.208663 | 0.122511 |
| | MSE | 6.243952 | 42.41076 | 2.794604 | MSE | 1.453568 | 19.58916 | 1.381536 |
| MPS | Bias | 1.680944 | 2.46109 | 0.327444 | Bias | 0.591814 | 1.354937 | 0.161187 |
| | MSE | 8.333252 | 44.09193 | 3.651898 | MSE | 1.839232 | 21.92581 | 1.698615 |
| LSE | Bias | 0.337176 | 0.769023 | 0.081114 | Bias | 0.239378 | 0.405721 | 0.044964 |
| | MSE | 0.539617 | 6.541684 | 0.880223 | MSE | 0.279419 | 2.966477 | 0.518669 |
| WLSE | Bias | 0.437686 | 1.043156 | 0.116244 | Bias | 0.296363 | 0.528444 | 0.059113 |
| | MSE | 0.85452 | 10.77633 | 1.208418 | MSE | 0.405863 | 4.601454 | 0.666113 |
| PE | Bias | 0.976288 | 2.056061 | 0.029938 | Bias | 0.630759 | 2.240984 | 0.042397 |
| | MSE | 2.756773 | 36.62246 | 1.091939 | MSE | 1.992241 | 40.09295 | 1.129633 |
| | n = 100 | | | | n = 300 | | | |
| MLE | Bias | 0.771949 | 1.543055 | 0.160362 | Bias | 0.12158 | 0.80965 | 0.09743 |
| | MSE | 2.921254 | 25.72089 | 1.882993 | MSE | 0.20373 | 10.40272 | 0.88347 |
| MPS | Bias | 1.059836 | 1.669283 | 0.204111 | Bias | 0.14244 | 0.95882 | 0.12739 |
| | MSE | 4.474894 | 28.8205 | 2.275575 | MSE | 0.24921 | 12.86450 | 1.09661 |
| LSE | Bias | 0.27465 | 0.479312 | 0.050318 | Bias | 0.14038 | 0.32109 | 0.04316 |
| | MSE | 0.327075 | 3.797982 | 0.621231 | MSE | 0.16372 | 1.96643 | 0.37393 |
| WLSE | Bias | 0.388808 | 0.629964 | 0.069554 | Bias | 0.13139 | 0.42016 | 0.05812 |
| | MSE | 0.608648 | 5.90818 | 0.836142 | MSE | 0.17866 | 2.94994 | 0.46487 |
| PE | Bias | 0.708949 | 2.446648 | 0.048272 | Bias | 0.35397 | 2.38321 | 0.10739 |
| | MSE | 2.245572 | 43.03393 | 1.184109 | MSE | 1.13059 | 38.40531 | 1.25764 |

**Table.6 The MSE and BIAS for different estimates of the LNH distribution with different values**

| Estimation | parm | α = 0.9 | β = 1.7 | λ = 2 | parm | α = 0.9 | β = 1.7 | λ = 2 |
|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | | n = 150 | | | |
| MLE | Bias | 1.077475 | 2.309147 | 0.224001 | Bias | 0.235602 | 0.994046 | 0.098974 |
| | MSE | 9.548177 | 88.58781 | 1.37842 | MSE | 0.973982 | 32.49193 | 0.478773 |
| MPS | Bias | 1.52575 | 2.74883 | 0.312691 | Bias | 0.361047 | 1.217011 | 0.142958 |
| | MSE | 13.3383 | 112.6092 | 1.891631 | MSE | 1.683738 | 43.87294 | 0.708559 |
| LSE | Bias | 0.588411 | 1.123808 | 0.107173 | Bias | 0.303019 | 0.542196 | 0.060359 |
| | MSE | 2.23715 | 19.1187 | 0.51515 | MSE | 0.789763 | 6.744807 | 0.260763 |
| WLSE | Bias | 0.677901 | 1.34998 | 0.126737 | Bias | 0.260243 | 0.572598 | 0.064027 |
| | MSE | 3.146453 | 28.23972 | 0.634019 | MSE | 0.760585 | 8.333041 | 0.268867 |
| PE | Bias | 1.536274 | 2.162142 | 0.059034 | Bias | 0.808543 | 2.500032 | 0.014497 |
| | MSE | 10.55131 | 63.1215 | 0.707779 | MSE | 4.652092 | 67.59279 | 0.869434 |
| | n = 100 | | | | n = 300 | | | |
| MLE | Bias | 0.521898 | 1.56595 | 0.150527 | Bias | 0.082806 | 0.340718 | 0.041075 |
| | MSE | 3.283916 | 58.46077 | 0.836739 | MSE | 0.164558 | 7.695085 | 0.165054 |
| MPS | Bias | 0.811676 | 1.73783 | 0.200792 | Bias | 0.123931 | 0.383874 | 0.056317 |
| | MSE | 5.380803 | 66.65809 | 1.100255 | MSE | 0.225051 | 9.650466 | 0.218034 |
| LSE | Bias | 0.397763 | 0.779637 | 0.088331 | Bias | 0.182143 | 0.32318 | 0.038309 |
| | MSE | 1.182111 | 11.15774 | 0.385945 | MSE | 0.389433 | 3.398002 | 0.155515 |
| WLSE | Bias | 0.393477 | 0.923255 | 0.102025 | Bias | 0.119471 | 0.276183 | 0.034995 |
| | MSE | 1.390203 | 16.22352 | 0.457169 | MSE | 0.267075 | 3.04711 | 0.133463 |
| PE | Bias | 1.077184 | 2.479614 | 0.002934 | Bias | 0.433132 | 2.579345 | 0.068891 |
| | MSE | 6.955389 | 71.64537 | 0.831405 | MSE | 2.154259 | 72.11955 | 0.956934 |

**Table.7 The MSE and BIAS for different estimates of the LNH distribution with different values**

| Estimation | parm | α = 1.2 | β = 2 | λ = 2.5 | parm | α = 1.2 | β = 2 | λ = 2.5 |
|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | | n = 150 | | | |
| MLE | Bias | 2.048544 | 2.584501 | 0.23496 | Bias | 0.622489 | 0.98395 | 0.097447 |
| | MSE | 32.31082 | 139.3048 | 2.016919 | MSE | 7.275323 | 38.83006 | 0.659646 |
| MPS | Bias | 2.481319 | 2.475248 | 0.293049 | Bias | 0.832926 | 1.105539 | 0.120776 |
| | MSE | 36.82994 | 130.3489 | 2.36914 | MSE | 8.141455 | 49.79035 | 0.825268 |
| LSE | Bias | 0.447452 | 0.822037 | 0.085451 | Bias | 0.254086 | 0.471465 | 0.055207 |
| | MSE | 2.141436 | 15.27939 | 0.522638 | MSE | 1.036212 | 6.461698 | 0.281876 |
| WLSE | Bias | 0.607032 | 1.109656 | 0.116071 | Bias | 0.284597 | 0.604212 | 0.071086 |
| | MSE | 3.841619 | 26.04498 | 0.701835 | MSE | 1.433278 | 10.16354 | 0.361773 |
| PE | Bias | 0.929867 | 3.039848 | 0.108791 | Bias | 0.353617 | 3.121066 | 0.203658 |
| | MSE | 9.25213 | 133.9141 | 0.930158 | MSE | 4.751573 | 127.9829 | 1.19168 |
| | n = 100 | | | | n = 300 | | | |
| MLE | Bias | 1.184055 | 1.405043 | 0.130911 | Bias | 0.214544 | 0.320322 | 0.032726 |
| | MSE | 14.74861 | 65.68415 | 1.086904 | MSE | 1.079685 | 9.411566 | 0.221032 |
| MPS | Bias | 1.582826 | 1.407527 | 0.159611 | Bias | 0.32885 | 0.287066 | 0.03831 |
| | MSE | 19.54527 | 67.86216 | 1.253358 | MSE | 1.805999 | 9.50374 | 0.245948 |
| LSE | Bias | 0.330301 | 0.611735 | 0.072778 | Bias | 0.176109 | 0.280537 | 0.033161 |
| | MSE | 1.473181 | 9.715541 | 0.409583 | MSE | 0.596345 | 3.420197 | 0.173005 |
| WLSE | Bias | 0.443582 | 0.735379 | 0.086148 | Bias | 0.182102 | 0.261039 | 0.030256 |
| | MSE | 2.410766 | 14.31737 | 0.505934 | MSE | 0.660381 | 3.725273 | 0.1685 |
| PE | Bias | 0.732264 | 2.742931 | 0.140044 | Bias | 0.151266 | 3.124965 | 0.236574 |
| | MSE | 7.902465 | 114.8551 | 1.019721 | MSE | 2.549177 | 131.3992 | 1.369404 |

**Table.8 The MSE and BIAS for different estimates of the LNH distribution with different values**

| Estimation | parm | α = 1.5 | β = 1.2 | λ =1.7 | parm | α = 1.5 | β = 1.2 | λ =1.7 |
|---|---|---|---|---|---|---|---|---|
| | n = 50 | | | | n = 150 | | | |
| MLE | Bias | 1.261234 | 0.711381 | 0.193167 | Bias | 0.448107 | 0.2182 | 0.036413 |
| | MSE | 31.75934 | 13.69002 | 0.901951 | MSE | 6.760806 | 2.118448 | 0.126457 |
| MPS | Bias | 2.08034 | 0.898023 | 0.276312 | Bias | 0.718639 | 0.154253 | 0.0494 |
| | MSE | 54.33612 | 21.81064 | 1.246787 | MSE | 11.00418 | 3.068382 | 0.167521 |
| LSE | Bias | 0.684094 | 1.366326 | 0.120121 | Bias | 0.402178 | 0.503662 | 0.050081 |
| | MSE | 7.81408 | 14.47074 | 0.339786 | MSE | 3.169429 | 3.542477 | 0.134185 |
| WLSE | Bias | 0.761294 | 1.535991 | 0.136809 | Bias | 0.368159 | 0.401837 | 0.041625 |
| | MSE | 10.97051 | 19.75357 | 0.419214 | MSE | 3.482594 | 3.238734 | 0.11597 |
| PE | Bias | 1.276058 | 2.691017 | 0.118891 | Bias | 0.458773 | 2.809125 | 0.203946 |
| | MSE | 27.93761 | 45.04055 | 0.632478 | MSE | 10.5055 | 45.35784 | 0.808819 |
| | n = 100 | | | | n = 300 | | | |
| MLE | Bias | 0.76266 | 0.553253 | 0.081685 | Bias | 0.169144 | 0.082635 | 0.009967 |
| | MSE | 14.96444 | 9.046802 | 0.314395 | MSE | 1.119374 | 0.685829 | 0.036673 |
| MPS | Bias | 1.233162 | 0.447662 | 0.113765 | Bias | 0.273288 | 0.022872 | 0.012322 |
| | MSE | 24.77268 | 9.044879 | 0.436126 | MSE | 2.013819 | 0.676378 | 0.039449 |
| LSE | Bias | 0.557835 | 0.710482 | 0.068445 | Bias | 0.246013 | 0.223297 | 0.02055 |
| | MSE | 4.891234 | 5.904778 | 0.203016 | MSE | 1.345576 | 1.393142 | 0.067892 |
| WLSE | Bias | 0.577662 | 0.729071 | 0.07229 | Bias | 0.176916 | 0.143822 | 0.014259 |
| | MSE | 6.639498 | 7.44816 | 0.211996 | MSE | 0.999635 | 0.850889 | 0.048241 |
| PE | Bias | 0.80343 | 2.546393 | 0.145503 | Bias | 0.209403 | 2.592269 | 0.233742 |
| | MSE | 17.21377 | 41.98287 | 0.69574 | MSE | 6.137387 | 39.49215 | 0.892119 |

## -Analysis of Data

The tabulated findings above reveal several noteworthy observations:

The Mean Square Errors (MSEs) exhibit a decreasing trend with an increase in sample size.

Upon comparing various estimation methods, it is evident that the Least Squares Estimation (LSE) method consistently yields superior results for estimating the parameters θ and λ.

Regarding the performance ranking of estimators for θ, the order from best to worst is LSE, Maximum Likelihood Estimation (MLE), and Weighted Least Squares Estimation (WLSE). Similarly, for λ, the ranking is LSE and WLSE. These insights provide valuable information about the efficiency and reliability of the different estimation methods under consideration.

## 10.Order statistics

Order statistics play a crucial role in non-parametric statistics and inference, serving as fundamental tools. The probability density function of the ith order statistic, denoted as what, in a random sample of size n drawn from the distribution proposed distribution is provided by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)\, F(x)^{i-1}\, [1 - F(x)]^{n-i}.$$

Using the binomial expansion, $f_{i:n}(x)$ can be reduced to:

$$f_{i:n}(x) = \sum_{m=0}^{\infty} t_{m+i} v_{m+i}(x;\lambda,\varphi),$$

where

$$t_{m+i} = \frac{(-1)^m n!}{(m+i)m!(n-i-m)!(i-1)!},$$

and $v_{m+i}(x) = (m+i)\, f(x;\lambda,\varphi) F(x;\lambda,\varphi)^{m+i-1}$ is the LNH density function with power parameter $m+i$. So, $f_{i:n}(x)$ is a linear combination of LNH densities.

$f_{i:n}(x)$ , the i th order statistic of LNH distribution is given by:

$$f_{i:n}(x) = \sum_{m=0}^{\infty} t_{m+i}\alpha\beta\lambda(1+\beta x)^{\alpha-1}[(1+\beta x)^{\alpha}-1]^{-\lambda-1}[1+\{(1+\beta x)^{\alpha} - 1\}^{-\lambda}]^{-2}[1+\{(1+\beta x)^{\alpha}-1\}^{-\lambda}]^{1-m-i},$$

**Table. 9. MLE Estimation under type-I censoring scheme. n=25,50**

| | n= 25 | | | n= 50 | | |
|---|---|---|---|---|---|---|
| | a = 1.2 | b =2 | λ = 2.5 | a = 1.2 | b =2 | λ = 2.5 |
| **Time of censoring (%) 30** | | | | | | |
| Bias | 0.649668 | 4.926434 | 0.60923 | 0.36796 | 4.786571 | 0.48561 |
| MSE | 9.937075 | 252.6143 | 4.964276 | 6.458349 | 224.4373 | 3.006853 |
| CP | 92.99 | 94.9 | 97.6 | 94.2 | 94.79 | 98 |
| **Time of censoring (%) 60** | | | | | | |
| Bias | 1.178346 | 5.06619 | 0.522384 | 1.008053 | 4.300733 | 0.368676 |
| MSE | 16.54518 | 270.9567 | 3.948463 | 11.66566 | 217.9167 | 2.455779 |
| CP | 93.6 | 97.3 | 97.9 | 94.5 | 94.3 | 97.5 |
| **Time of censoring (%) 90** | | | | | | |
| Bias | 1.883939 | 4.843844 | 0.466951 | 1.722957 | 3.148915 | 0.279291 |
| MSE | 31.26156 | 275.6015 | 3.870508 | 23.25247 | 173.1045 | 2.28072 |
| CP | 92.7 | 97.5 | 98.1 | 92.7 | 91.5 | 96 |
| **Time of censoring (%) 99.99** | | | | | | |
| Bias | 3.02046 | 3.259536 | 0.33233 | 1.749798 | 2.193454 | 0.208957 |
| MSE | 48.89261 | 198.5864 | 3.257292 | 24.06855 | 119.4353 | 1.908957 |
| CP | 93.17 | 92.4 | 96.4 | 93.1 | 90.3 | 94.3 |

**Table. 10. MLE Estimation under type-I censoring scheme. n=75,100**

| | n= 75 | | | n= 100 | | |
|---|---|---|---|---|---|---|
| | a = 1.2 | b =2 | λ = 2.5 | a = 1.2 | b =2 | λ = 2.5 |
| **Time of censoring (%) 30** | | | | | | |
| Bias | 0.427144 | 4.447768 | 0.385735 | 0.445588 | 3.944125 | 0.327314 |
| MSE | 5.237462 | 212.1619 | 2.153511 | 4.732963 | 181.4919 | 1.743009 |
| CP | 93.3 | 94.1 | 97.9 | 94.1 | 93.67 | 97.5 |
| **Time of censoring (%) 60** | | | | | | |
| Bias | 0.926301 | 4.224687 | 0.338311 | 0.765755 | 3.309591 | 0.268514 |
| MSE | 11.02658 | 212.048 | 2.096333 | 6.995857 | 150.6801 | 1.586327 |
| CP | 95.1 | 94.6 | 98 | 92.7 | 93.6 | 96.6 |
| **Time of censoring (%) 90** | | | | | | |
| Bias | 1.165455 | 2.634399 | 0.220286 | 0.673126 | 2.350916 | 0.220129 |
| MSE | 12.92989 | 133.8139 | 1.681694 | 7.016888 | 103.9434 | 1.421902 |
| CP | 93.99 | 90.5 | 94.8 | 94.4 | 92.3 | 94 |
| **Time of censoring (%) 99.99** | | | | | | |
| Bias | 1.355785 | 1.874996 | 0.182617 | 1.124943 | 1.679063 | 0.147696 |
| MSE | 18.51045 | 91.40019 | 1.458482 | 13.49209 | 80.17721 | 1.174893 |
| CP | 93.6 | 92.1 | 91.9 | 93.3 | 92.4 | 92 |

**Table. 11. MLE Estimation under type-I censoring scheme. n=25,50**

| | n= 25 | | | n= 50 | | |
|---|---|---|---|---|---|---|
| | a = 0.7 | b =1.5 | λ = 3 | a = 0.7 | b =1.5 | λ = 3 |
| Time of censoring (%) 30 | | | | | | |
| Bias | 0.287533 | 5.119673 | 0.590579 | 0.253107 | 4.465618 | 0.473634 |
| MSE | 1.794882 | 110.2303 | 6.307976 | 1.410951 | 91.98796 | 4.279184 |
| CP | 94 | 98.2 | 99 | 95 | 98.4 | 97.9 |
| Time of censoring (%) 60 | | | | | | |
| Bias | 0.58528 | 4.069458 | 0.472594 | 0.567348 | 3.365379 | 0.351993 |
| MSE | 2.833577 | 77.881 | 4.933781 | 1.885304 | 62.20015 | 3.404274 |
| CP | 94.5 | 98.7 | 98.1 | 94.1 | 99.1 | 98.8 |
| Time of censoring (%) 90 | | | | | | |
| Bias | 1.119589 | 3.401549 | 0.400968 | 0.829607 | 2.764668 | 0.309253 |
| MSE | 5.3892 | 62.79687 | 4.35316 | 3.299092 | 48.62101 | 3.218999 |
| CP | 93.7 | 98.4 | 98.1 | 95 | 97.6 | 99.4 |
| Time of censoring (%) 99.99 | | | | | | |
| Bias | 1.752244 | 2.571771 | 0.28201 | 1.052161 | 2.231204 | 0.238137 |
| MSE | 7.787667 | 51.35312 | 3.764009 | 3.828153 | 41.67167 | 2.858282 |
| CP | 93.1 | 97.1 | 98.2 | 94.7 | 96.8 | 98.6 |

**Table. 12. MLE Estimation under type-I censoring scheme. n=75,100**

| | n= 75 | | | n= 100 | | |
|---|---|---|---|---|---|---|
| | a = 0.7 | b =1.5 | λ = 3 | a = 0.7 | b =1.5 | λ = 3 |
| Time of censoring (%) 30 | | | | | | |
| Bias | 0.30419 | 3.878214 | 0.373808 | 0.214556 | 3.482151 | 0.33849 |
| MSE | 1.065539 | 79.1608 | 3.128471 | 0.811364 | 67.30136 | 2.731215 |
| CP | 94.5 | 98.5 | 98.9 | 94.5 | 97.8 | 98.4 |
| Time of censoring (%) 60 | | | | | | |
| Bias | 0.466985 | 2.854431 | 0.298177 | 0.398496 | 2.745943 | 0.277565 |
| MSE | 1.40391 | 50.85458 | 2.636724 | 1.164411 | 48.28711 | 2.438393 |
| CP | 95.1 | 98 | 99 | 94.6 | 98.1 | 98.9 |
| Time of censoring (%) 90 | | | | | | |
| Bias | 0.644375 | 2.431623 | 0.247722 | 0.436571 | 2.304572 | 0.247457 |
| MSE | 2.012367 | 43.16993 | 2.525183 | 1.54027 | 37.62998 | 2.311934 |
| CP | 93.9 | 98.2 | 99.2 | 94 | 97.3 | 99.6 |
| Time of censoring (%) 99.99 | | | | | | |
| Bias | 0.623556 | 1.943982 | 0.204284 | 0.531608 | 1.778257 | 0.190045 |
| MSE | 1.885022 | 33.45876 | 2.219914 | 1.651643 | 29.28222 | 1.986813 |
| CP | 94.5 | 96.3 | 98 | 95 | 95.5 | 97.8 |

**Table. 13. MLE Estimation under type-II censoring scheme. n=25,50**

| | n= 25 | | | n= 50 | | |
|---|---|---|---|---|---|---|
| | a = 1.2 | b =2 | λ = 2.5 | a = 1.2 | b =2 | λ = 2.5 |
| **Time of censoring (%) 30** | | | | | | |
| **Bias** | 2.084741 | 2.551816 | 0.493283 | 1.604249 | 2.984072 | 0.363348 |
| **MSE** | 24.63218 | 130.9428 | 3.626362 | 16.60021 | 143.9592 | 2.100432 |
| **CP** | 94.7 | 92.6 | 97.1 | 94.9 | 92.8 | 97.2 |
| **Time of censoring (%) 60** | | | | | | |
| **Bias** | 2.572791 | 3.222154 | 0.414318 | 1.896524 | 3.464295 | 0.320033 |
| **MSE** | 35.84274 | 171.0519 | 3.066956 | 23.69069 | 177.5908 | 2.155219 |
| **CP** | 94.1 | 93.1 | 97.6 | 94.6 | 92.8 | 97.4 |
| **Time of censoring (%) 90** | | | | | | |
| **Bias** | 3.000692 | 3.359583 | 0.334707 | 2.060485 | 2.704569 | 0.251161 |
| **MSE** | 49.23146 | 194.3237 | 2.944403 | 27.84157 | 143.3958 | 2.023756 |
| **CP** | 93.1 | 92.8 | 96.5 | 93.9 | 91.4 | 95.6 |
| **Time of censoring (%) 99.99** | | | | | | |
| **Bias** | 2.861621 | 3.051454 | 0.312589 | 1.798465 | 2.479311 | 0.233297 |
| **MSE** | 50.42647 | 178.2683 | 3.042994 | 28.99403 | 129.3269 | 1.989734 |
| **CP** | 91.3 | 91.2 | 96.3 | 91.8 | 91 | 94.3 |

**Table. 14. MLE Estimation under type-II censoring scheme. n=100, 75**

| | n= 100 | | | n= 75 | | |
|---|---|---|---|---|---|---|
| | a = 1.2 | b =2 | λ = 2.5 | a = 1.2 | b =2 | λ = 2.5 |
| **Time of censoring (%) 30** | | | | | | |
| **Bias** | 1.35176 | 3.079558 | 0.297115 | 1.327378 | 2.663802 | 0.241741 |
| **MSE** | 12.50296 | 145.6913 | 1.638181 | 10.96793 | 123.6279 | 1.269024 |
| **CP** | 95.2 | 92.8 | 97.1 | 95.1 | 92.6 | 96.8 |
| **Time of censoring (%) 60** | | | | | | |
| **Bias** | 1.671424 | 2.935675 | 0.253907 | 1.441978 | 2.574743 | 0.21595 |
| **MSE** | 17.07926 | 146.5809 | 1.674127 | 13.54311 | 119.2073 | 1.345503 |
| **CP** | 95.7 | 92.3 | 96.3 | 94.7 | 92.2 | 95.5 |
| **Time of censoring (%) 90** | | | | | | |
| **Bias** | 1.675032 | 2.547646 | 0.224217 | 1.259407 | 1.973118 | 0.174527 |
| **MSE** | 22.64548 | 128.519 | 1.70453 | 15.23907 | 91.61826 | 1.296694 |
| **CP** | 93.7 | 91.8 | 93.7 | 93.5 | 92.5 | 93 |
| **Time of censoring (%) 99.99** | | | | | | |
| **Bias** | 1.625018 | 1.536477 | 0.147488 | 1.232104 | 1.14944 | 0.105827 |
| **MSE** | 22.95176 | 72.99336 | 1.264841 | 14.85007 | 51.08049 | 0.914206 |
| **CP** | 92.4 | 91.6 | 91.6 | 93.7 | 93.4 | 91.9 |

**Table. 15. MLE Estimation under type-II censoring scheme. n=25,50**

| | n= 25 | | | n= 50 | | |
|---|---|---|---|---|---|---|
| | a = 0.7 | b =1.5 | λ = 3 | a = 0.7 | b =1.5 | λ = 3 |
| **Time of censoring (%) 30** | | | | | | |
| Bias | 1.816447 | 2.730621 | 0.499595 | 1.480692 | 2.758956 | 0.361672 |
| MSE | 7.265226 | 60.94906 | 5.902773 | 5.044278 | 59.43761 | 3.28162 |
| CP | 94.1 | 94.4 | 98.2 | 95 | 95.6 | 98.3 |
| **Time of censoring (%) 60** | | | | | | |
| Bias | 2.194529 | 2.606784 | 0.334392 | 1.623432 | 2.632783 | 0.294152 |
| MSE | 10.28213 | 51.98401 | 3.562091 | 6.396382 | 48.97948 | 2.920658 |
| CP | 93.8 | 96.7 | 98.8 | 94.8 | 97.3 | 98.5 |
| **Time of censoring (%) 90** | | | | | | |
| Bias | 2.36241 | 2.799715 | 0.316273 | 1.618352 | 2.413075 | 0.257377 |
| MSE | 12.79473 | 55.87062 | 3.902538 | 7.504111 | 43.31698 | 2.964913 |
| CP | 92.1 | 97.4 | 98.5 | 93.8 | 98.2 | 98.8 |
| **Time of censoring (%) 99.99** | | | | | | |
| Bias | 2.298949 | 2.398637 | 0.259408 | 1.450915 | 2.192598 | 0.232355 |
| MSE | 13.15698 | 47.4648 | 3.625283 | 7.475958 | 39.2485 | 2.741396 |
| CP | 92.4 | 96.8 | 98.6 | 93.1 | 97.1 | 98.9 |

**Table. 16. MLE Estimation under type-II censoring scheme. n=100, 75**

| | n= 100 | | | n= 75 | | |
|---|---|---|---|---|---|---|
| | a = 0.7 | b =1.5 | λ = 3 | a = 0.7 | b =1.5 | λ = 3 |
| **Time of censoring (%) 30** | | | | | | |
| Bias | 1.254841 | 2.629106 | 0.28642 | 1.143134 | 2.511262 | 0.265355 |
| MSE | 3.684308 | 52.9904 | 2.473688 | 3.175733 | 48.12521 | 2.312511 |
| CP | 94.2 | 95.5 | 97.9 | 94.6 | 95 | 98.5 |
| **Time of censoring (%) 60** | | | | | | |
| Bias | 1.206677 | 2.496348 | 0.276813 | 1.218319 | 2.05215 | 0.211143 |
| MSE | 4.444362 | 43.98303 | 2.581589 | 4.100263 | 34.98315 | 2.064838 |
| CP | 94.2 | 97.3 | 99.2 | 93.8 | 96.1 | 99.1 |
| **Time of censoring (%) 90** | | | | | | |
| Bias | 1.334895 | 2.052657 | 0.213719 | 1.101739 | 1.755144 | 0.180545 |
| MSE | 5.605285 | 36.08629 | 2.428999 | 4.380492 | 29.58061 | 2.013796 |
| CP | 93.7 | 96 | 99 | 93.1 | 95.4 | 98.4 |
| **Time of censoring (%) 99.99** | | | | | | |
| Bias | 0.910486 | 1.942925 | 0.213947 | 0.595874 | 1.690169 | 0.183638 |
| MSE | 3.949319 | 32.8467 | 2.28624 | 2.268045 | 27.6933 | 1.973701 |
| CP | 94.7 | 95.9 | 98.5 | 95.2 | 94.6 | 98.2 |

## Conclusion

- the Empirical Mean Residual Life value needs to significantly increase with sample size.
- the Renyi entropy value increases, the level of uncertainty in the system also grows.
-Upon comparing various estimation methods, it is evident that the Least Squares Estimation (LSE) method consistently yields superior results for estimating the parameters θ and λ.

## References

1. Almarashi A M, Algarni A, Okasha H, Nassar M. (2022a). On reliability estimation of Nadarajah–Haghighi distribution under adaptive type-I progressive hybrid censoring scheme. Qual Reliab Eng Int.; 38: 817–833.
2. Cícero R. B. Dias, Morad Alizadeh and Gauss M. Cordeiro. (2018a) The beta Nadarajah-Haghighi distribution. Hacettepe Journal of Mathematics and Statistics. Volume 47 (5) , 1302 - 1320
   DOI: https://doi.org/10.2298/YJOR190415027M
3. Fernando A. Peña-Ramírez, Renata Rojas Guerraa, Diego Ramos Canterleb, Gauss M. Cordeiroc (2020a): "The logistic Nadarajah–Haghighi distribution and its associated regression model for reliability applications", Reliability Engineering and System Safety 204 ,107196.
4. Hilary I. Okagbue, Member, IAENG, Pelumi E. Oguntunde, Abiodun A. Opanuga & Sheila A. Bishop (2018b): "Marshall-Olkin Nadarajah-Haghighi distribution: ordinary differential equations". Proceedings of The World Congress on Engineering and Computer Science, vol |, WCECS 2018, 23-25 October, San Francisco, U.S.A.
5. Hisham A.H. Elsayed & Haitham M. Yousof (2019a): "The Burr X Nadarajah Haghighi Distribution: Statistical Properties and Application to the Exceedances of Flood Peaks Data", Journal of Mathematics and Statistics 2019, Volume 15: 146.157. DOI: 10.3844/jmssp.2019.146.157
6. J. J. Swain, S. Venkatraman, J.R. Wilson, Least-squares estimation of distribution functions in johnson’s translation system, J. Stat. Comput. Simul., 29 (1988),271–297.
7. Kao, j. H. (1958) computer methods for estimating weibull parameters in reliability studies, trans. IRE Reliab. Qual. Control 13, pp. 15-22.
8. Kao, j. H. (1959) A graphical estimation of mixed weibull parameters in life testing electron tube, technometrics 1, pp. 389-407
9. Kenney, j. F. and E. S. (1962). Mathematics of statistics. 3rd ed. Princeton, NJ: Chapman and Hall, pp. 101-102.

10. Liuquan Sun., Zhigang Zhang (2010). A Class of Transformed Mean Residual Life Models with Censored Survival Data., 104(486): 803–815. doi:10.1198/jasa.2009.0130.
11. M. H. Tahir, Gauss M. Cordeiro, Ayman Alzaatreh, M. Mansoor & M. Zubair (2016): The Logistic-X Family of Distributions and Its Applications, Communications in Statistics - Theory and Methods, DOI: 10.1080/03610926.2014.980516
12. Marija MINIC, (2020b): Estimation parameters of Nadarajah-Haghighi extension Exponential distribution using perfect and imperfect Ranked set Sample", Yugoslav Journal of Operations Research, 30(2020), Number2, 177-198.
13. M. A. Aljarrah, Carl Lee and Felix Famoye. (2014): On generating T-X family of distributions using quantile functions Journal of Statistical Distributions and Applications, 1:2
14. Moors, J. J. (1988). A quantile alternative for kurtosis. J. Royal Statist. Soc. D, vol. 37, pp. 25-32.
15. Morad Alizadehl, Mahdi Rasekhi, Haitham M. Yousof, Thiago G. Ramires & G. G. Hamedani (2018c): Extended Exponentiated Nadarajah-Haghighi Model: Mathematical Properties, Characterizations and Applications, Studia Scientiarum Mathematicarum Hungarica 55 (4), 498-522. DOI:10.1556/012.2018.55.4.1408
16. Ms. Sana, M. Faizan (2019b): Bayesian Estimation for Nadarajah-Haghighi Distribution Based on Upper Record Values, Pak.j.stat.oper.res. Vol.XV No.1 pp217-230
17. Muhammad Ahsan-ul-Haq, Junaid Ahmed , Mohammed Albassam & Muhammad Aslam (2022b) " Power Inverted Nadarajah–Haghighi Distribution: Properties, Estimation, and Applications", Hindawi, Journal of Mathematics, Volume 2022, Article ID 9514739, 10 pages , https://doi.org/10.1155/2022/9514739
18. Nadarajah, S., and Haghighi, F., (2011) "An extension of the exponential distribution", Statistics, 45(6) 543-558.
19. Robert M. Gray. (2023). Entropy and Information Theory. Springer-Verlag, New York
20. Sanku Dey, Liang Wang & Mazen Nassar (2021): Inference on Nadarajah–Haghighi distribution with constant stress partially accelerated life tests under progressive type-II censoring, Journal of Applied Statistics. DOI: 10.1080/02664763.2021.1928014
    url: http://www.stat.washington.edu/jaw/
21. Ziyad A. Alhussain & Essam A. Ahmed (2017): Estimation of Exponentiated Nadarajah-Haghighi Distribution Under Progressively Type-II Censored Sample with Application to Bladder Cancer Data, Indian J. Pure Appl. Math., 51(2) :631-657, June, DOI:10.1007/s13226-020-0421-9

# تقدير معالم توزيع لوجيستك نادراج - حقيقي Logistic Nadarajah-Haghighi

**د. محمد محمد عبد القادر؛ د. تامر حسن محمد**

**الملخص**

تقدم الدراسة توزيع نادراج حقيقي اللوجيستي كإمتداد لتوزيع Nadarajah-Haghighi الذي اقترحه **Haghighi**وآخرون في عام ٢٠١١، ، مما يوفر بديلا قيما للتوزيعات التقليدية مثل توزيعات ، .gamma, Weibull, exponentiated exponential.

تعرض هذه الدراسة لأربع طرق لتقدير معالم توزيع نادراج حقيقي اللوجستي Logistic Nadarajah-Haghighi (LNH) هي: الامكان الأعظم (MLE) والمربعات الصغري(LS) والمربعات الصغري الموزونة(WLS) بالاضافة لطريقة التقدير المئوي(PCE)، وذلك باستخدام العينات الكاملة تارة والعينات المراقبة أخري بناءً على نوعين من الرقابة: | & ||. تقوم الدراسة بالمفاضلة بين هذه الطرق باستخدام التحيز التربيعي والتباينات لهذه التقديرات من خلال عمليات محاكاة مونت كارلو. بالإضافة إلى ذلك، فإن الدراسة تعرض لخصائص التوزيع وبعض المقاييس الأخري مثل متوسط الحياة المتبقية، وريني انتروبيا.

**الكلمات المفتاحية:** دالة الوسيط – العزوم - متوسط الحياة المتبقية - ريني انتروبيا – الاحصاءات الترتيبية