A Bivariate Discrete Lindley Distribution and Applications

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A Bivariate Discrete Lindley Distribution and Applications

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Abstract

In this paper a bivariate discrete Lindley distribution has been derived from a discrete Lindely distribution using Farlie-Gumbel-Morgenstern copula. Some properties of this distribution such as probability generating function, conditional distributions, Pearson's correlation andreliability parameter are studied. To estimate the parameters of the distribution, three methods of estimation were presented. Method of moments, maximum likelihood estimation and two-step maximum likelihood. Finally, simulation study and a practical application were made on real data to show the appropriateness of the proposed distribution on these data.

Keywords: Lindely distribution, Bivariate distribution, Discrete distribution, Maximum likelihood estimation, Method of moments, Farlie-Gumbel-Morgenstern copula.

1. Introduction

The bivariate discrete data analysis is very common in many of practical applications. It arises in many real-life situations. For example, the number of insurance claims for two different reasons or the number of goals scored by two competing teams are a typical example of bivariate discrete data.

Kundu and Nekoukhou (2018), Barbiero (2017), Nekoukhou and Kundu (2017), Ong and Ng (2013), Johnson et al. (1997) and Kocherlakota and Kocherlakota (1992) have introduced different methods of the bivariate discrete data analysis.

Recently, Lee and Cha (2015) introduced two general methods, (a) minimization and (b) maximization, with the aim of creating a class of bivariate discrete distributions. They detailed some special cases such as the bivariate Poisson distribution, the bivariate geometric distribution, the bivariate negative binomial distribution and the bivariate binomial distribution. Although this method can produce a very flexible class of bivariate discrete distributions, the jointly probability mass function (PMF) may not be in a simplified form in many cases.

For this reason, the process of estimating unknown parameters becomes difficult in many cases. Another point, the bivariate discrete distributions suggested by Lee and Cha (2015) may not have the same corresponding marginal distributions. For example, the bivariate discrete Poisson distribution they proposed does not contain the marginal Poisson distribution.

Chakraborty and Chakraborty (2015) presented various methods for generating discrete data from analogues from continuous probability distributions. Among these methods, if the primary continuous failure time X has a survival function S(x) = P(X > x), then the probability mass function (pmf) can be written for the discrete random variable associated with this continuous distribution

$$P(X = x) = P_x = S(x) - S(x + 1); \quad x = 0, 1, 2, \ldots$$

This method, which was introduced to create a new discrete distribution, has been applied recently to generate many discrete distributions. For example, Chakraborty and Chakravarthy (2012, 2014, 2015) examined the discrete version of gamma, Gumbel and power distributions, Nekoukhou,et al. (2013) presented discrete generalized Lindley distribution of a second type. Gómez-Déniz and Calderin (2011) analyzed the discrete Lindely distribution, Gómez-Déniz (2010) derived a new generalization of the geometric distribution using Marshall-Olkin scheme, Krishna and Singh (2009) analyzed the discrete Burr distribution, Kemp (2008) constructed the discrete half-normal distribution and Roy (2004) derived the discrete Rayleigh distribution.

Using the method presented by Chakraborty (2015), a discrete Lindley distribution with a probability mass function (PMF) can be writtenas follows

$$P(X = x) = \left(1 + \frac{\theta x}{\theta + 1}\right) e^{-\theta x} - \left(1 + \frac{\theta + \theta x}{\theta + 1}\right) e^{-\theta(x+1)}$$

$$= \left(\frac{\theta}{\theta + 1}\right)\left(1 - e^{-\theta}\right)(x + c_1)e^{-\theta x} \; ; \; x = 0, 1, 2, \ldots, \infty \qquad (1)$$

where $c_1 = \frac{(1-e^{-\theta}) + \theta(1-2e^{-\theta})}{\theta(1-e^{-\theta})}$

With cumulative distribution function (CDF) as follows,

$$F_x(x, \theta) = P(X \le x) = \sum_{i=0}^{x} \left(\frac{\theta}{\theta + 1}\right)\left(1 - e^{-\theta}\right)(1 + c_1)e^{-\theta i}$$

$$= 1 - \left(1 + \frac{\theta + \theta x}{\theta + 1}\right) e^{-\theta(x+1)} \; ; \quad x = 0, 1, 2, \ldots, \infty \quad (2)$$

The previous result can be easily provenusing the finite geometric series, then the survival function will be,

$$S_x(x, \theta) = P(X > x) = 1 - F_x(x, \theta) = \left(1 + \frac{\theta + \theta x}{\theta + 1}\right) e^{-\theta(x+1)} \; ; \quad x = 0, 1, 2, \ldots, \infty$$

Regarding bivariate discretedistributions, Barbiero (2017) presented a discrete Weibull distribution using Farlie-Gumbel-Morgenstern (FGM) copula, which is the same capula that we will use to construct the bivariate distribution in this paper.

This paper is outlined as follows, In Section 2, the proposed bivariate discrete Lindleydistribution by used Farlie-Gumbel-Morgenstern (FGM) copula and its properties are presented and discussed. Moreover, The methods of moments, maximum likelihood estimation and two-step maximum likelihoodare determined in Section 3; Section 4 presents a simulation study assessing the performance of the estimators andcomparison between the three methods of estimation; whereas in Section 5 an application to real data is provided ; Finally, in Section 6, some concluding remarks are given.

2. The Bivariate Discrete Lindley Distribution

In this section, we introduce the bivariate discrete Lindleydistribution, by specifying itscdf andpmf, and then derive some mathematical properties.

2.1. Definition

A bivariate discrete Lindley distribution (BDL) can be obtained by linking together two discrete Lindley distributions $F_x(x,\theta), F_y(y,\alpha)$ via the Farlie-Gumbel-Morgenstern (FGM) copula(Farlie 1960) with parameter$-1 \leq \beta \leq 1$.The bivariate FGM copula is givenby

$C(u,v) = uv\,[1 + \beta(1-u)(1-v)],\ u,v \in [0,1]$via the parameter β.

If $\beta > 0$, the FGM copula provides positive dependence; if $\beta < 0$, it returns negative dependence; when β is zero, it reduces to the independence copula.

Fredricks and Nelsen (2007) drived the formula for Spearman's and Kendall's correlation coefficient as follows,

$$\rho_{\text{Spearman}} = 12\int_0^1\int_0^1 C(u,v)du\,dv - 3 = \frac{\beta}{3}$$

$$\rho_{\text{Kendall}} = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 = \frac{2\beta}{9}$$

and $\frac{-\beta}{3} \leq \rho_{\text{Spearman}} \leq \frac{-\beta}{3}, \frac{-2\beta}{9} \leq \rho_{\text{Kendall}} \leq \frac{-2\beta}{9}$.

A bivariate distribution with discrete Lindley (BDL) margins $X\sim F_x(x,\theta)$ and $Y\sim F_y(y,\alpha)$, related by the FGM copula can be built by simply defining its bivariate survival function as $S(x,y) = C(S_x(x), S_y(y)),\ x,y = 0,1,2,\dots,\infty$ from which we get:

$$S(x,y) = S_x(x)S_y(y)\left[1 + \beta\left(F_x(x)\right)\left(F_y(y)\right)\right],$$

$$= \left(1+\frac{\theta+\theta x}{\theta+1}\right)\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\theta(x+1)-\alpha(y+1)}$$

$$\left[1 + \beta\left(1-\left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right)\left(1-\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha(y+1)}\right)\right], \alpha,\theta > 0 \quad (3)$$

and bivariate cdf given by

$$F(x,y) = 1 - S_x(x) - S_y(y) + S(x,y)$$

$$= \left(1-\left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right)\left(1-\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha(y+1)}\right)$$

$$\left[1 + \beta\left(1+\frac{\theta+\theta x}{\theta+1}\right)\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\theta(x+1)-\alpha(y+1)}\right], \alpha,\theta > 0 \quad (4)$$

The corresponding bivariate pmf is then given (see also Barbiero 2017) by recalling the relationship between bivariate pmf and cdf:

$$P(x,y) = S(x,y) - S(x-1,y) - S(x,y-1) + S(x-1,y-1)$$

$$= P(x)P(y)\left[1 + \beta\left\{(2F_x(x) - P(x) - 1)(2F_y(y) - P(y) - 1)\right\}\right] \quad (5)$$

$$P(x,y) = \left(\frac{\theta\alpha}{(\theta+1)(1+\alpha)}\right)(x+c_1)(y+c_2)(1-e^{-\theta})(1-e^{-\alpha})e^{-\theta x-\alpha y}$$

$$\left[1 + \beta\left(1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x} - \left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right)\left(1-\left(1+\frac{\alpha y}{\alpha+1}\right)e^{-\alpha y} - \left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha(y+1)}\right)\right] \quad (6)$$

where $c_2 = \frac{(1-e^{-\alpha})+\alpha(1-2e^{-\alpha})}{\alpha(1-e^{-\alpha})}$

Figures 1,2 show the 3-dimatintions plots for the pmf and cdf of BDL distribution with different

value of $\theta, \alpha\ and\ \beta$.

$\boldsymbol{\theta = 0.3, \alpha = 1.5 \text{ and } \beta = 0.5}$ $\boldsymbol{\theta = 0.5, \alpha = 0.3 \text{ and } \beta = -0.5}$

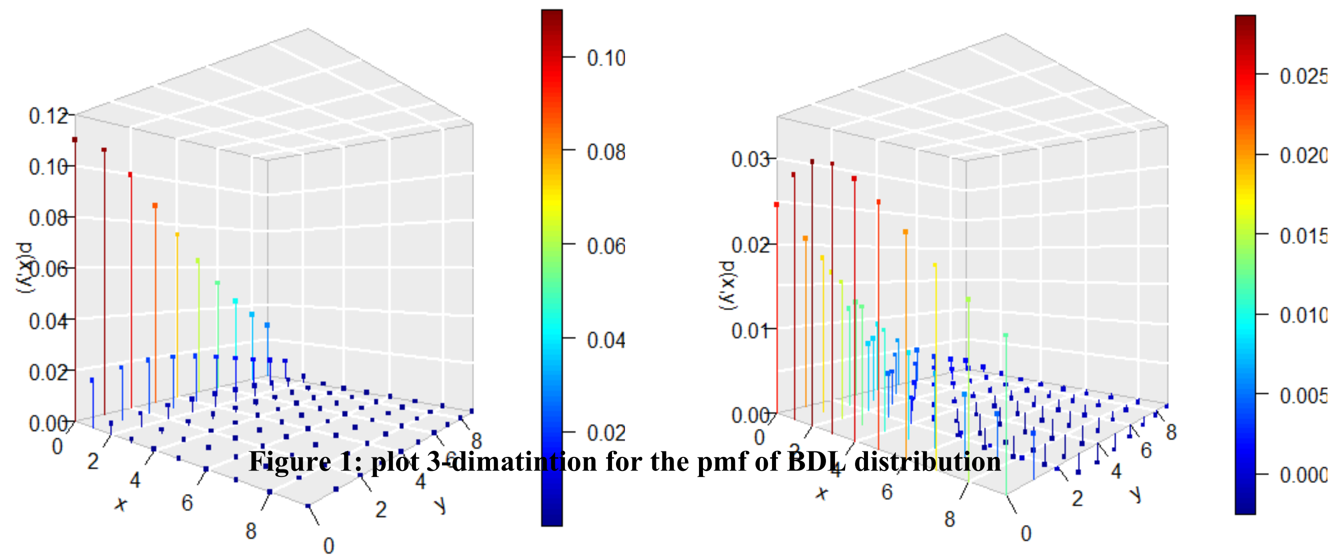

Figure 1: plot 3-dimatintion for the pmf of BDL distribution

$\boldsymbol{\theta = 0.3, \alpha = 1.5 \text{ and } \beta = 0.5}$ $\boldsymbol{\theta = 0.5, \alpha = 0.3 \text{ and } \beta = -0.5}$

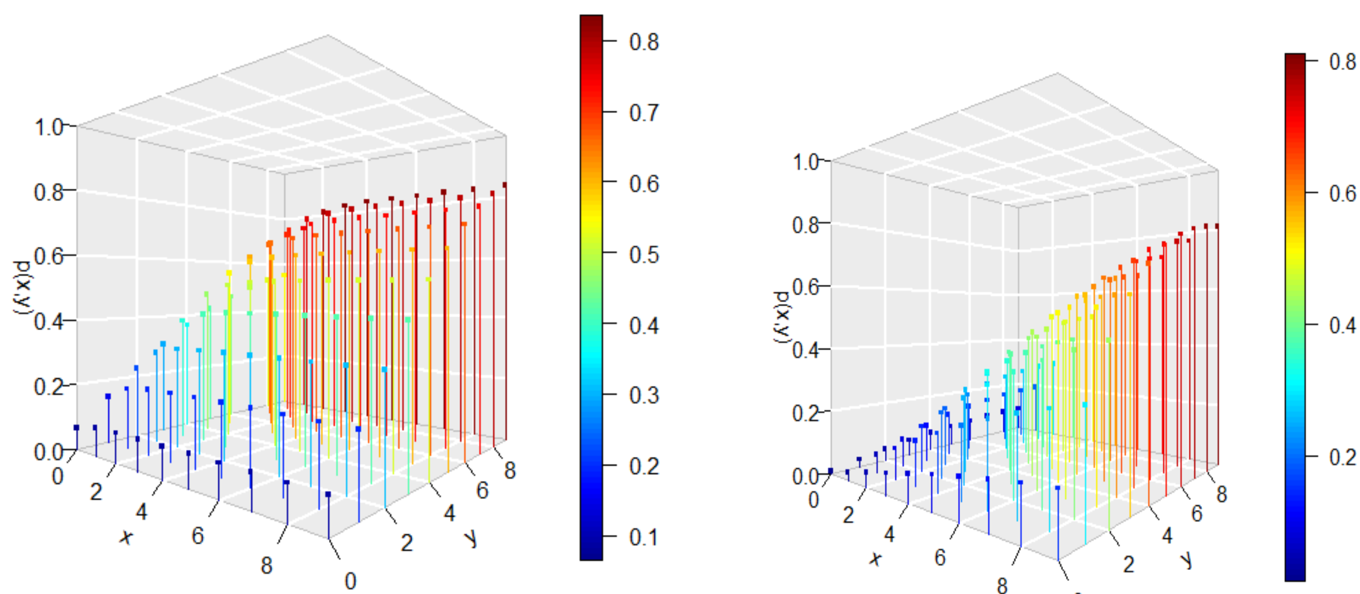

Figure 2: plot 3-dimatintion for the cdf of BDL distribution

2.2. The Bivariate Failure Rate

The bivariate failure rate can be defined as $r(x,y) = p(x,y)/S(x,y)$, which assumes the following expression for the BDL distribution r.v., for $x = 0,1,\dots; y = 0,1,\dots$:

$$r(x,y) = \frac{\left(\frac{\theta\alpha}{(\theta+1)(1+\alpha)}\right)(x+c_1)(y+c_2)\left(1-e^{-\theta}\right)\left(1-e^{-\alpha}\right)}{\left(1+\frac{\theta+\theta x}{\theta+1}\right)\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\theta-\alpha}}$$

$$\times \frac{\left[\left[1+\beta\left(1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}-\left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right)\left(1-\left(1+\frac{\alpha y}{\alpha+1}\right)e^{-\alpha y}-\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha(y+1)}\right)\right]\right]}{\left[1+\beta\left(1-\left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right)\left(1-\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha(y+1)}\right)\right]} \quad (7)$$

Proposition 2.1. for the bivariate failure rate$r(x,y)$,

If $\beta = 0$ (x and y are independence), $r(x,y)$is constant and equal to

$$r_0 = \frac{\left(\frac{\theta}{\theta+1}\right)(x+c_1)\left(1-e^{-\theta}\right)}{\left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta}} \times \frac{\left(\frac{\alpha}{\alpha+1}\right)(y+c_2)\left(1-e^{-\alpha}\right)}{\left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha}}.$$

And, for $\beta \neq -1$,we get

$$\lim_{(x,y)\to(\infty,\infty)} r(x,y) = \frac{\left(1-e^{-\theta}\right)\left(1-e^{-\alpha}\right)}{e^{-\theta-\alpha}}\ , \quad (8)$$

$$\lim_{(x,y)\to(0,0)} r(x,y) = \frac{\alpha\,\theta\,c_1\,c_2\left(1-e^{-\theta}\right)\left(1-e^{-\alpha}\right)}{(2\theta+1)(2\alpha+1)e^{-\theta-\alpha}} \times \frac{1+\beta\left[\left(\frac{2\theta+1}{\theta+1}\right)\left(\frac{2\alpha+1}{\alpha+1}\right)e^{-\theta-\alpha}\right]}{1+\beta\left[\left(1-\left(\frac{2\theta+1}{\theta+1}\right)e^{-\theta}\right)\left(1-\left(\frac{2\alpha+1}{\alpha+1}\right)e^{-\alpha}\right)\right]}, \quad (9)$$

then

$$\frac{\left(1-e^{-\theta}\right)\left(1-e^{-\alpha}\right)}{e^{-\theta-\alpha}} \le r(x,y) \le \frac{\alpha\,\theta\,c_1\,c_2\left(1-e^{-\theta}\right)\left(1-e^{-\alpha}\right)}{(2\theta+1)(2\alpha+1)e^{-\theta-\alpha}} \times \frac{1+\beta\left[\left(\frac{2\theta+1}{\theta+1}\right)\left(\frac{2\alpha+1}{\alpha+1}\right)e^{-\theta-\alpha}\right]}{1+\beta\left[\left(1-\left(\frac{2\theta+1}{\theta+1}\right)e^{-\theta}\right)\left(1-\left(\frac{2\alpha+1}{\alpha+1}\right)e^{-\alpha}\right)\right]}$$

2.3. Probability Generating Function

Let X and Y be a pair of integer-valued random variables with joint distribution $P(X = x, Y = y)$,then the bivariate probability generating function is given by $G(t,s) = E(t^x s^y)$. For the BDL distribution, the bivariate probabilitygenerating function is as follows:

$$G(t,s)=\frac{q_1q_2\left(1-e^{-\theta}\right)(1-e^{-\alpha})\left[(1-c_1)te^{-\theta}+c_1\right]\left[(1-c_2)se^{-\alpha}+c_2\right]}{[(1-te^{-\theta})(1-se^{-\alpha})]^2}$$
$$+\beta\, q_1q_2\left(1-e^{-\theta}\right)(1-e^{-\alpha})\left\{\left(\frac{(1-c_1)te^{-\theta}+c_1}{(1-te^{-\theta})^2}-\frac{\varphi_1^te^{-\theta}+q_1\left(1-e^{-\theta}\right)\varphi_2^t}{(1-te^{-2\theta})^2}\right)\left(\frac{(1-c_2)se^{-\alpha}+c_2}{(1-se^{-\alpha})^2}-\frac{\varphi_1^se^{-\alpha}+q_2(1-e^{-\alpha})\varphi_2^s}{(1-se^{-2\alpha})^2}\right)\right\} \qquad (10)$$

where

$\varphi_1^t=\left(q_1-c_1+1\right)t^2e^{-4\theta}+\left[c_1\left(3q_1-1\right)+1\right]te^{-2\theta}+c_1,$

$\varphi_2^t=\left(c_1{}^2-3c_1+2\right)t^2e^{-4\theta}-\left(2c_1{}^2-3c_1-1\right)te^{-2\theta}+c_1{}^2,$

$\varphi_1^s=\left(q_2-c_2+1\right)s^2e^{-4\alpha}+\left[c_2\left(3q_2-1\right)+1\right]se^{-2\alpha}+c_2,$

$\varphi_2^s=\left(c_2{}^2-3c_2+2\right)s^2e^{-4\alpha}-\left(2c_2{}^2-3c_2-1\right)se^{-2\alpha}+c_2{}^2,$

$q_1=\frac{\theta}{\theta+1}$ and $q_2=\frac{\alpha}{\alpha+1}$.

Proof

The bivariate probability generating function is defined as:

$$G(t,s)=E(t^xs^y)=\sum_{x=0}^{\infty}\sum_{y=0}^{\infty}t^xs^y\,P(x,y)$$

Using equation (6), we obtain

$$G(t,s)=E(t^xs^y)$$
$$=E(t^x)E(s^y)+\beta E(t^x[2F_x(x)-P(x)-1])E\left(s^y\left[2F_y(y)-P(y)-1\right]\right)$$

Substituting from 1 and 2 and using the properties of the sum of the finite geometric series, we can get the result mentioned in 10.

The probability generating functions of the marginal distributions $\mathrm{P(X = x)}$ and $\mathrm{P(Y = y)}$ are $\mathrm{G(t,1) = E(t^x)}$ and $\mathrm{G(1,s) = E(s^y)}$, respectively.

2.4. Conditional distributions

The conditional distribution of $Y|\,X=x$ is

$$P_{y|x}(y,x) = \frac{P(x,y)}{P(x)}$$
$$= \left(\frac{\alpha}{(1+\alpha)}\right)(y+c_2)(1-e^{-\alpha})e^{-\alpha y}\left[1 + \beta\left(1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x} - \left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right)\left(1-\left(1+\frac{\alpha y}{\alpha+1}\right)e^{-\alpha y} - \left(1+\frac{\alpha+\alpha y}{\alpha+1}\right)e^{-\alpha(y+1)}\right)\right] \qquad (11)$$

Moreover, the conditional expected value of $Y|\ X = x$ can be then easily computed and is equal to

$$E(Y|\ X = x) = E(Y) + \beta([2F_x(x) - P(x) - 1])E\left(\left[2F_y(Y) - P(Y) - 1\right]\right)$$
$$= \frac{(1+2\alpha)e^{-\alpha} - (1+\alpha)e^{-2\alpha}}{(1+\alpha)(1-e^{-\alpha})^2} + \beta\left(\frac{(1+2\alpha)e^{-\alpha} + \alpha e^{-2\alpha} + (3\alpha-1)e^{-3\alpha} - (2+6\alpha)e^{-4\alpha} - (4\alpha+1)e^{-5\alpha} + (\alpha+2)e^{-6\alpha} + (1+\alpha)e^{-7\alpha}}{(1+\alpha)(1-e^{-\alpha})^3(1+e^{-\alpha})^4}\right)$$
$$\left(1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x} - \left(1+\frac{\theta+\theta x}{\theta+1}\right)e^{-\theta(x+1)}\right) \qquad (12)$$

The conditional c.d.f $P(Y \geq y, |X = x)$, of $Y|\ X = x$ can be derived easily and written as:

$$P(Y \leq y, |X = x) = F_y(y,\alpha) + \beta F_y(y,\alpha)S_y(y,\alpha)\{1 - p(x) - 2F_x(x-1,\theta)\} \qquad (13)$$

Note that ,we can get the same results for $X|\ Y = y$ using the same method.

2.5. Simulation

In order to simulate a sample from the BDL distribution with parameters θ and α we can use the following steps :

1. Generate a random pair (u_1, u_2) from two independent uniform r.v.s in (0,1), $u_1 \sim \text{Unif}(0,1)$ and $u_2 \sim \text{Unif}(0,1)$;
2. Put $x = F_x^{-1}(u_1)$ where $F_x^{-1}(.)$ denotes the quantile functions of discrete Lindley distribution with parameter θ, i.e., $x = \frac{-\ln(1-u_1)}{\theta}$, with [.] indicating the ceiling function.
3. Put $P(Y \geq y, |X = x) = u_2$, we get $z = \frac{2\,u_2}{a+b}$

   where $a = 1 + \beta\{1 - p(x) - 2F_x(x-1,\theta)\}$, $b = [a^2 - 4(1-a)u_2]^{\frac{1}{2}}$ and $z = F_y(y,\alpha)$

4. After solving aquadratic equation, put $y = F_y^{-1}(z)$ where $F_y^{-1}(.)$ denotes the quantile functions of discrete Lindley distribution with parameters α, i.e., $y = \frac{-\ln(1-z)}{\alpha}$, with [.] indicating the ceiling function.
5. (x, y) is a random pair from the BDL distribution.

2.6. Pearson's Correlations

For a discrete Lindley distribution with parameter θ, the expected value is written as:

$$E(X) = \frac{(1+2\theta)e^{-\theta} - (1+\theta)e^{-2\theta}}{(1+\theta)(1-e^{-\theta})^2}$$

The variance is equal to

$$V(X) = \frac{(1+2\theta)(1+\theta)e^{-\theta} - (2\theta^2+4\theta+2)e^{-2\theta} + (1+\theta)e^{-3\theta}}{(1+\theta)(1-e^{-\theta})^4},$$

and the covariance $Cov(X,Y)$ between the two margins havethe followingexpression:

$$Cov(X,Y) = \beta E([2F_x(X) - P(X) - 1])E([2F_y(Y) - P(Y) - 1])$$

$$= \beta\left(\frac{(1+2\theta)e^{-\theta} + \theta e^{-2\theta} + (3\theta-1)e^{-3\theta} - (2+6\theta)e^{-4\theta} - (4\theta+1)e^{-5\theta} - (\theta+2)e^{-6\theta} + (1+\theta)e^{-7\theta}}{(1+\theta)(1-e^{-\theta})^3(1+e^{-\theta})^4}\right)$$

$$\left(\frac{(1+2\alpha)e^{-\alpha} + \alpha e^{-2\alpha} + (3\alpha-1)e^{-3\alpha} - (2+6\alpha)e^{-4\alpha} - (4\alpha+1)e^{-5\alpha} + (\alpha+2)e^{-6\alpha} + (1+\alpha)e^{-7\alpha}}{(1+\alpha)(1-e^{-\alpha})^3(1+e^{-\alpha})^4}\right) \quad (14)$$

Then, Pearson's correlation coefficientρ_{xy} between the two margins can be easily calculated as;

$$\rho_{xy} = \frac{cov(XY)}{\sqrt{V(X)V(Y)}} \qquad (15)$$

From (14), if the value of $\beta = 0$, then the value of the correlation coefficient ρ_{xy} is equal to zero, which means the independence of the two variables, if the value of $\beta > 0$, this indicates that the correlation is positive, and if the value of $\beta < 0$ it is an negativerelation.

2.7. Reliability parameter

In reliability context inferences about R=P(Y<X), where X and Y have independent distributions, are a subject of interest. For example in mechanical reliability of a system, if X is the strength of a component which is subject to stress Y, then R is a measure of system performance. The system fails, if at any time the applied stress is greate than its

strength. Stress- strength reliability has been discussed in Kapur and Lamberson (1977). However, in some real life cases, strength or stress can have discrete distribution, for example, when the strength is the number of the products that factory produces and the stress is the number of the products that customers want to buy (Jovanović 2017).

For the BDL distribution, the stress-strength parameter $R = P(Y < X)$ can be computed as

$R = P(Y < X) = \sum_{x=0}^{\infty} \sum_{y=0}^{x} P(x,y)$ with $P(x,y)$ given by equation3, and has the following result:

$$R = \sum_{y=0}^{\infty} P(y)\, F(y,\theta) + \beta \sum_{y=0}^{\infty} P(y)\, F(y,\theta)\, S(y,\theta)\{F(y,\alpha) - P(y) - 1\} \qquad (16)$$

From the previous relationship, the value of R can be easily obtained, since these sums are the sum of infinite geometric sequences, but their expansion will be a large number of terms, and therefore it can be calculated numerically easily as shown in Table 1.

Table (1) The value of R at different values of the distribution parameters

$\boldsymbol{\beta}$	$\boldsymbol{\theta = 0.3, \alpha = 0.4}$	$\boldsymbol{\theta = 0.4, \alpha = 0.3}$	$\boldsymbol{\theta = 1.3, \alpha = 0.4}$	$\boldsymbol{\theta = 1.5, \alpha = 1.6}$	$\boldsymbol{\theta = 1.6, \alpha = 1.5}$
-0.6	0.653	0.424	0.212	0.744	0.709
-0.3	0.65	0.433	0.225	0.755	0.72
0	0.648	0.441	0.237	0.766	0.731
0.3	0.646	0.45	0.25	0.777	0.743
0.6	0.644	0.459	0.263	0.787	0.754

It is very obvious that the value of R increases with the increasing value of the parameter β for all combinations of parameters θ and α. From the Figure 3, it becomes clear that the relationship between both R and β is a linear relationship

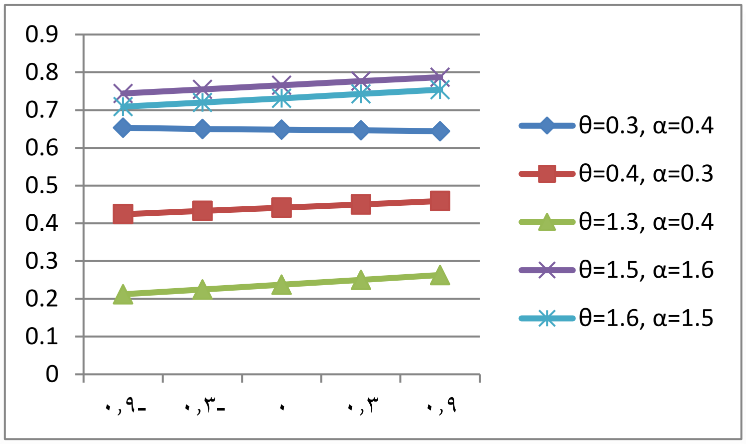

Figure 3: The relationship between both R and β

3. Estimation

Several methods for estimating the parameters $\theta, \alpha$and β of the BDE distribution of equation 3 can be envistigated, given a random sample $(x_i, y_i), i = 1,2, \dots, n$. Two versions of the method of moments are here considered as well as the maximum likelihood method.

3.1. Method of Moments

A method of moments (MoM) is presented, derived by equating the marginals moments with the mixed moment to the corresponding sample quantities. Denoting by $\hat{\mu}_{xy} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$ the sample mixed moment.

It is possible to reach the estimations of moments $(\breve{\theta}, \breve{\alpha})$ by solving the following equations numerically

$$\bar{x} = \frac{(1+2\theta)e^{-\theta} - (1+\theta)e^{-2\theta}}{(1+\theta)(1-e^{-\theta})^2}$$

$$\bar{y} = \frac{(1+2\alpha)e^{-\alpha} - (1+\alpha)e^{-2\alpha}}{(1+\alpha)(1-e^{-\alpha})^2}$$

Then, by substituting into the following function, the moment estimator for β . can be obtained

$$\breve{\beta} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}}{\left[E([2F_x(X) - P(X) - 1])\, E\left([2F_y(Y) - P(Y) - 1]\right)\right]\Big|_{\theta = \breve{\theta}, \alpha = \breve{\alpha}}}$$

3.2. Maximum Likelihood Method

From (3), the log-likelihood function $\mathfrak{l}(\theta,\alpha,\beta)$ is given by:

$$\mathfrak{l}(\theta,\alpha,\beta) = n[ln(\alpha) + ln(\theta) - ln(1+\alpha) - ln\,(1+\theta)] + nln\left(1-e^{-\theta}\right) + nln(1-e^{-\alpha}) - \theta\sum_{i=1}^{n} x_i - \alpha\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} ln(x_i + c_1) + \sum_{i=1}^{n} ln(y_i + c_2) + \phi(\theta,\alpha,\beta,x_i,y_i) \quad (17)$$

where

$$\phi(\theta,\alpha,\beta,x_i,y_i) = \sum_{i=1}^{n} ln\left[1 + \beta\left(1-\left(1+\frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i} - \left(1+\frac{\theta+\theta x_i}{\theta+1}\right)e^{-\theta(x_i+1)}\right)\left(1 - \left(1+\frac{\alpha y_i}{\alpha+1}\right)e^{-\alpha y_i} - \left(1+\frac{\alpha+\alpha y_i}{\alpha+1}\right)e^{-\alpha(y_i+1)}\right)\right]$$

The Maximum Likelihood estimates (MLEs) of θ, α and β are obtained by maximizing (17). The derivatives of (17) with respect to the unknown parameters are given as,

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - \frac{n}{\theta+1} + \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{\grave{c}_1}{x_i + c_1} + \frac{\partial\phi(\theta,\alpha,\beta,x_i,y_i)}{\partial\theta} \quad (18)$$

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - \frac{n}{\alpha+1} + \frac{ne^{-\alpha}}{1-e^{-\alpha}} - \sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\frac{\grave{c}_2}{y_i + c_2} + \frac{\partial\phi(\theta,\alpha,\beta,x_i,y_i)}{\partial\alpha} \quad (19)$$

$$\frac{\partial l}{\partial\beta} = \frac{\partial\phi(\theta,\alpha,\beta,x_i,y_i)}{\partial\beta} \quad (20)$$

where

$\grave{c}_1 = \frac{e^{-\theta}}{\left(1-e^{-\theta}\right)^2} - \frac{1}{\theta^2}$ and $\grave{c}_2 = \frac{e^{-\alpha}}{(1-e^{-\alpha})^2} - \frac{1}{\alpha^2}$

The likelihood equations are given as

$$\frac{\partial l}{\partial\theta} = 0, \frac{\partial l}{\partial\alpha} = 0 \text{ and } \frac{\partial l}{\partial\beta} = 0,$$

gives the maximum likelihood estimators $\hat{\varphi} = (\hat{\theta}, \hat{\alpha}, \hat{\beta})^{\grave{}}$ of $\varphi = (\theta,\alpha,\beta)^{\grave{}}$. As $n \to \infty$ the asymptotic distribution of the MLE$(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ for BDE Distribution is given as

$$\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix} \sim N\left[\begin{pmatrix}\theta\\ \alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\hat{v}_{11} & \hat{v}_{12} & \hat{v}_{13}\\ \hat{v}_{21} & \hat{v}_{22} & \hat{v}_{23}\\ \hat{v}_{31} & \hat{v}_{32} & \hat{v}_{33}\end{pmatrix}\right] \quad (21)$$

where

$$v^{-1} = -E\begin{bmatrix}\frac{\partial^2 l}{\partial\theta^2} & \frac{\partial^2 l}{\partial\theta\partial\alpha} & \frac{\partial^2 l}{\partial\theta\partial\beta}\\ \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta}\\ \frac{\partial^2 l}{\partial\beta\partial\theta} & \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta^2}\end{bmatrix}. \quad (22)$$

In relation to the asymptotic variance-covariance matrix of the ML estimators of the parameters, it can be approximated by numerically inverting the above Fisher's information matrix F. Thus, the approximate $100(1-\gamma)\,\%$ two-sided confidence intervals for α, θ and β can be, respectively, easily obtained by.

$\hat{\theta} \pm Z_{\gamma/2}\sigma_{\hat{\theta}}$, $\hat{\alpha} \pm Z_{\gamma/2}\sigma_{\hat{\alpha}}$ and $\hat{\beta} \pm Z_{\gamma/2}\sigma_{\hat{\beta}}$

where Z_{γ} is the γ^{th} upper percentile of the standard normal distribution.

3.3. Two-Step Maximum Likelihood

Since the traditional Maximum Likelihood (ML) method discussed above can be computationally cumbersome, the literature has suggested the two-step ML method (TSML).

According to this method, which is proposed in Joe and Xu (1996) and Joe (1997) and is also called "Inference Functions for Margins" (IFM), first, the parameters of the marginal distributions are estimated based on the ML function of each marginal distribution, then the multivariate parameters are estimated from the ML function of the multivariate distribution as a second step with the substitution of the estimates for the parameters of the marginal distributions from the first step.

While this method is asymptotically less efficient than the traditional ML method (see again Joe (1997)), this method has the distinct advantage of minimizing the dimensions of the problem. Which is particularly useful when it is needed to the numerical maximization.

For the bivariate distribution in this study, the TSML method is similar to the case presented in Barbiero (2017), when estimates the parameters for the marginal distributions of the two marginal distributions as if they were independent as follows,

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - \frac{n}{\theta+1} + \frac{ne^{-\theta}}{1-e^{-\theta}} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\grave{c}_1}{x_i + c_1} \quad (23)$$

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}-\frac{n}{\alpha+1}+\frac{ne^{-\alpha}}{1-e^{-\alpha}}-\sum_{i=1}^{n} y_i+\sum_{i=1}^{n}\frac{\grave{c}_2}{y_i+c_2} \qquad (24)$$

By solving each equation after equating by zero, we get the parameter estimates for the marginal distributions for the parameters of the two marginal distributions $\tilde{\theta}, \tilde{\alpha}$. Then the second step is to substitute those estimates into the ML function for the bivariate distribution and calculate the parameter value by setting the following equation to zero:

$$\frac{\partial l}{\partial \beta}=\frac{\partial \phi(\theta,\alpha,\beta,x_i,y_i)}{\partial \beta}\Big|_{\theta=\tilde{\theta},\alpha=\tilde{\alpha}} \qquad (20)$$

4. Simulation Study

In this section, a simulation is done for a comparison between the estimation methods, MoM, MLE and TSML, used for estimating BDL distribution parameters by R language.

The values of the parameters $\theta, \alpha and\ \beta$ are chosen as the following cases for the random variables using different sample size n =20, 50, 100 and 150,

Case1: $(\theta = 0.4\ , \alpha = 0.3\ , \beta = -0.6)$,Case2: $(\theta = 1.5\ , \alpha = 1.6\ , \beta = -0.6)$,

Case 3:$(\theta = 0.4\ , \alpha = 0.3\ , \beta = 0.7)$, andCase4: $(\theta = 1.5\ , \alpha = 1.6\ , \beta = 0.7)$

All computations are obtained. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculate the Bias, the MSE, and the length of confidence interval (L.CI) as follows,

$$Bias = \hat{\delta} - \delta$$

Where $\hat{\delta}$is the estimated value of δ .

$$MSE = Maen\left(\hat{\delta} - \delta\right)^2$$

and

$L.CI = Upper.CI - Lower.CI$

We restricted the number of repeated samples to 1000.

Table 2: Estimation of the Parameters of BDL Distribution: Case 1

n	methods		mean	Bias	MSE	L.CI
20	MLE	$\hat{\theta}$	0.523	-0.123	0.02	0.29
		$\hat{\alpha}$	0.181	0.119	0.02	0.285
		$\hat{\beta}$	-0.726	0.126	0.021	0.277
	MoM	$\breve{\theta}$	0.91	-0.51	0.338	1.093
		$\breve{\alpha}$	-0.19	0.49	0.322	1.122
		$\breve{\beta}$	-0.905	0.405	0.337	1.12
	TSML	$\tilde{\theta}$	0.567	-0.167	0.037	0.376
		$\tilde{\alpha}$	0.143	0.157	0.034	0.379
		$\tilde{\beta}$	-0.09	-0.51	0.34	1.11
50	MLE	$\hat{\theta}$	0.45	-0.05	0.003248	0.109
		$\hat{\alpha}$	0.25	0.05	0.0033	0.109
		$\hat{\beta}$	-0.651	0.051	0.00343	0.112
	MoM	$\breve{\theta}$	0.604	-0.204	0.055	0.446
		$\breve{\alpha}$	0.092	0.208	0.056	0.451
		$\breve{\beta}$	-0.799	0.199	0.052	0.431
	TSML	$\tilde{\theta}$	0.563	-0.163	0.036	0.383
		$\tilde{\alpha}$	0.126	0.174	0.04	0.376
		$\tilde{\beta}$	-0.087	-0.513	0.347	1.138
100	MLE	$\hat{\theta}$	0.426	-0.026	0.000865	0.057
		$\hat{\alpha}$	0.276	0.024	0.000788	0.057
		$\hat{\beta}$	-0.626	0.026	0.000879	0.056
	MoM	$\breve{\theta}$	0.502	-0.102	0.014	0.233
		$\breve{\alpha}$	0.199	0.101	0.014	0.232
		$\breve{\beta}$	-0.702	0.102	0.014	0.222
	TSML	$\tilde{\theta}$	0.56	-0.16	0.035	0.373
		$\tilde{\alpha}$	0.129	0.171	0.039	0.377
		$\tilde{\beta}$	-0.131	-0.469	0.296	1.088
150	MLE	$\hat{\theta}$	0.423	-0.023	0.000698	0.052
		$\hat{\alpha}$	0.274	0.026	0.000899	0.058
		$\hat{\beta}$	-0.627	0.027	0.000957	0.061
	MoM	$\breve{\theta}$	0.503	-0.103	0.014	0.241
		$\breve{\alpha}$	0.193	0.107	0.015	0.225
		$\breve{\beta}$	-0.697	0.097	0.013	0.22
	TSML	$\tilde{\theta}$	0.555	-0.155	0.033	0.371
		$\tilde{\alpha}$	0.13	0.17	0.038	0.375
		$\tilde{\beta}$	-0.07	-0.53	0.354	1.057

Table 3: Estimation of the Parameters of BDL Distribution: Case 2

n	methods		mean	Bias	MSE	L.CI
20	MLE	$\hat{\theta}$	1.623	-0.123	0.021	0.285
		$\hat{\alpha}$	1.478	0.122	0.02	0.282
		$\hat{\beta}$	-0.474	-0.126	0.021	0.284
	MoM	$\breve{\theta}$	2.005	-0.505	0.331	1.08
		$\breve{\alpha}$	1.117	0.483	0.318	1.142
		$\breve{\beta}$	-0.988	0.388	0.316	1.097
	TSML	$\tilde{\theta}$	1.664	-0.164	0.036	0.37
		$\tilde{\alpha}$	1.44	0.16	0.035	0.373
		$\tilde{\beta}$	-0.121	-0.479	0.317	1.157
50	MLE	$\hat{\theta}$	1.55	-0.05	0.003385	0.114
		$\hat{\alpha}$	1.549	0.051	0.003414	0.11
		$\hat{\beta}$	-0.549	-0.051	0.00345	0.114
	MoM	$\breve{\theta}$	1.707	-0.207	0.057	0.461
		$\breve{\alpha}$	1.404	0.196	0.052	0.451
		$\breve{\beta}$	-0.801	0.201	0.053	0.442
	TSML	$\tilde{\theta}$	1.653	-0.153	0.032	0.371
		$\tilde{\alpha}$	1.436	0.164	0.037	0.389
		$\tilde{\beta}$	-0.089	-0.511	0.342	1.115
100	MLE	$\hat{\theta}$	1.526	-0.026	0.000853	0.056
		$\hat{\alpha}$	1.575	0.025	0.0008166	0.055
		$\hat{\beta}$	-0.575	-0.025	0.0008369	0.056
	MoM	$\breve{\theta}$	1.602	-0.102	0.014	0.231
		$\breve{\alpha}$	1.495	0.105	0.014	0.229
		$\breve{\beta}$	-0.707	0.107	0.015	0.215
	TSML	$\tilde{\theta}$	1.66	-0.16	0.034	0.368
		$\tilde{\alpha}$	1.439	0.161	0.035	0.382
		$\tilde{\beta}$	-0.112	-0.488	0.322	1.137
150	MLE	$\hat{\theta}$	1.523	-0.023	0.000.416	0.058
		$\hat{\alpha}$	1.575	0.025	0.0008395	0.06
		$\hat{\beta}$	-0.574	-0.026	0.0008629	0.053
	MoM	$\breve{\theta}$	1.603	-0.103	0.014	0.241
		$\breve{\alpha}$	1.493	0.107	0.015	0.225
		$\breve{\beta}$	-0.697	0.097	0.013	0.22
	TSML	$\tilde{\theta}$	1.659	-0.159	0.035	0.383
		$\tilde{\alpha}$	1.416	0.184	0.044	0.383
		$\tilde{\beta}$	-0.109	-0.491	0.321	1.108

Table 4: Estimation of the Parameters of BDL Distribution: Case 3

n	methods		mean	Bias	MSE	L.CI
20	MLE	$\hat{\theta}$	0.526	-0.126	0.021	0.278
		$\hat{\alpha}$	0.173	0.127	0.021	0.281
		$\hat{\beta}$	0.571	0.129	0.022	0.283
	MoM	$\breve{\theta}$	0.905	-0.505	0.331	1.08
		$\breve{\alpha}$	-0.183	0.483	0.318	1.142
		$\breve{\beta}$	0.222	0.488	0.316	1.097
	TSML	$\tilde{\theta}$	0.561	-0.161	0.035	0.369
		$\tilde{\alpha}$	0.137	0.163	0.035	0.367
		$\tilde{\beta}$	0.225	0.475	0.304	1.096
50	MLE	$\hat{\theta}$	0.449	-0.049	0.003208	0.113
		$\hat{\alpha}$	0.249	0.051	0.00338	0.111
		$\hat{\beta}$	0.65	0.05	0.003353	0.114
	MoM	$\breve{\theta}$	0.601	-0.201	0.054	0.456
		$\breve{\alpha}$	0.103	0.197	0.053	0.457
		$\breve{\beta}$	0.903	-0.203	0.054	0.45
	TSML	$\tilde{\theta}$	0.561	-0.161	0.035	0.38
		$\tilde{\alpha}$	0.145	0.155	0.033	0.365
		$\tilde{\beta}$	0.219	0.481	0.305	1.067
100	MLE	$\hat{\theta}$	0.427	-0.027	0.0009258	0.054
		$\hat{\alpha}$	0.275	0.025	0.00085	0.06
		$\hat{\beta}$	0.673	0.027	0.000986	0.06
	MoM	$\breve{\theta}$	0.503	-0.103	0.014	0.235
		$\breve{\alpha}$	0.202	0.098	0.013	0.226
		$\breve{\beta}$	0.801	-0.101	0.014	0.233
	TSML	$\tilde{\theta}$	0.561	-0.161	0.036	0.383
		$\tilde{\alpha}$	0.135	0.165	0.036	0.372
		$\tilde{\beta}$	0.197	0.503	0.334	1.11
150	MLE	$\hat{\theta}$	0.425	-0.025	0.0008421	0.056
		$\hat{\alpha}$	0.276	0.024	0.0007933	0.056
		$\hat{\beta}$	0.675	0.025	0.000833	0.056
	MoM	$\breve{\theta}$	0.505	-0.105	0.015	0.233
		$\breve{\alpha}$	0.204	0.096	0.013	0.228
		$\breve{\beta}$	0.81	-0.11	0.015	0.214
	TSML	$\tilde{\theta}$	0.553	-0.153	0.034	0.397
		$\tilde{\alpha}$	0.129	0.171	0.037	0.342
		$\tilde{\beta}$	0.155	0.545	0.383	1.148

Table 5: Estimation of the Parameters of BDL Distribution: Case 4

n	methods		mean	Bias	MSE	L.CI
20	MLE	$\hat{\theta}$	1.629	-0.129	0.022	0.294
		$\hat{\alpha}$	1.473	0.127	0.022	0.289
		$\hat{\beta}$	0.578	0.122	0.02	0.29
	MoM	$\breve{\theta}$	1.99	-0.49	0.321	1.11
		$\breve{\alpha}$	1.089	0.511	0.349	1.164
		$\breve{\beta}$	0.151	0.549	0.383	1.116
	TSML	$\tilde{\theta}$	1.667	-0.167	0.038	0.39
		$\tilde{\alpha}$	1.433	0.167	0.037	0.381
		$\tilde{\beta}$	0.207	0.493	0.331	1.167
50	MLE	$\hat{\theta}$	1.549	-0.049	0.003261	0.113
		$\hat{\alpha}$	1.55	0.05	0.003339	0.114
		$\hat{\beta}$	0.649	0.051	0.003397	0.111
	MoM	$\breve{\theta}$	1.694	-0.194	0.051	0.448
		$\breve{\alpha}$	1.401	0.199	0.053	0.46
		$\breve{\beta}$	0.894	-0.194	0.05	0.433
	TSML	$\tilde{\theta}$	1.671	-0.171	0.038	0.374
		$\tilde{\alpha}$	1.425	0.175	0.04	0.368
		$\tilde{\beta}$	0.197	0.503	0.34	1.152
100	MLE	$\hat{\theta}$	1.525	-0.025	0.00082	0.057
		$\hat{\alpha}$	1.576	0.024	0.000784	0.055
		$\hat{\beta}$	0.675	0.025	0.000835	0.057
	MoM	$\breve{\theta}$	1.602	-0.102	0.014	0.228
		$\breve{\alpha}$	1.5	0.1	0.013	0.224
		$\breve{\beta}$	0.803	-0.103	0.014	0.221
	TSML	$\tilde{\theta}$	1.666	-0.166	0.037	0.374
		$\tilde{\alpha}$	1.433	0.167	0.037	0.378
		$\tilde{\beta}$	0.216	0.484	0.317	1.125
150	MLE	$\hat{\theta}$	1.524	-0.024	0.00078	0.058
		$\hat{\alpha}$	1.576	0.024	0.000781	0.057
		$\hat{\beta}$	0.676	0.024	0.000776	0.057
	MoM	$\breve{\theta}$	1.6	-0.1	0.013	0.224
		$\breve{\alpha}$	1.505	0.095	0.012	0.219
		$\breve{\beta}$	0.797	-0.097	0.013	0.233
	TSML	$\tilde{\theta}$	1.681	-0.181	0.04	0.338
		$\tilde{\alpha}$	1.446	0.154	0.033	0.368
		$\tilde{\beta}$	0.164	0.536	0.353	1.007

We note from Tables 2 through 5 that the parameters of the BDL distribution were estimated by the three methods that were previously explored. It was found from the results of the estimates that the ML method is the best estimation method, as its bias values are relatively small compared to the rest of the estimation methods. It also has the lowest values for both MSE and L.CI.

Although the MoM method gives relatively reasonable estimates, it sometimes gives very far values for the estimator β which affects the values of MSE and L.CI to be relatively large for this estimator. Because both θ and α are estimated separately, then using these estimators to estimate β, therefore the process of controlling the properties of the estimators together is not achieved. Despite the above, both ML and MoM achieve that the estimators have the property of consistency. Also, as it is noticed with the large sample size, the values of Bias and MSE decreased. Looking at the TSML method, we find that it suffers from the same thing as the MoM method, which is sometimes it gives very far values for the estimator β, which affects the values of MSE and L.CI. This is due to the same reason that the estimation process is carried out in two stages and it does not achieve the consistency property of the estimators where the values of Bias and MSE are not affected by the large sample size.

From the above, it can be said that the ML method gives estimators with good properties compared to Mom and TSML. Despite the expended effort in the estimation process with the two methods MoM and TSML is much less than the method of the ML.

5. Real Data Example

Table (6) shows the results that collected from a trusted soccer's site of the Italian League 2019/2020, where the variable X expresses the number of home team goals, and the variable Y expresses the number of away team goals and the numbers inside the cells express the repetitions of each result.

Table 6: Results of the Italian League 2019/2020

y\x	0	1	2	3	4	≥ 5	Total
0	69 (67.26)	49 (40.66)	24 (28.12)	15 (10.26)	4 (7.98)	2 (1.898)	163 (156.18)
1	51 (46.36	24 (29.26)	16 (15.20)	8 (7.22)	4 (3.334)	1 (1.462)	104 (102.84)
2	26 (26.98)	18 (17.48)	9 (9.50)	5 (4.56)	3 (2.076)	0 (0.913)	61 (61.51)
3	14 (14.44)	8 (9.50)	5 (5.32)	2 (2.537)	3 (1.162)	0 (0.512)	32 (33.47)
4	8 (7.60)	5 (4.94)	2 (2.723)	1 (1.332)	0 (1.14)	0 (0.000)	16 (17.74)
≥ 5	3 (3.701)	1 (2.51)	0 (1.372)	0 (0.672)	0 (0.000)	0 (0.000)	4 (8.26)
Total	171 (166.341)	105 (104.35)	56 (62.236)	31 (26.581)	14 (15.691)	3 (4.784)	380 (379.983)

We found that the sample statistics summary are, $\bar{x} = 1.4395$, $\bar{y} = 1.27$, $\mathrm{Var(x)} = 1.64$, $\mathrm{Var(y)} = 1.27 \hat{\rho}_{xy} = 0.014$.

Table7: Parameter estimates for the BDL Distribution applied to the Italian League 2019/2020

Method	θ	α	β
ML	0.858	0.994	0.105
MoM	1.118	1.431	0.198
TSML	0.915	1,099	0.177

The observed Fisher Information Matrix is given by:

$$\hat{\mathrm{I}} = \begin{pmatrix} 0.000613 & 0.000245 & -0.000325 \\ 0.000245 & 0.000641 & -0.00139 \\ -0.000325 & -0.00139 & 0.00219 \end{pmatrix}$$

The asymptotic standard errors of the three MLEs can be easily derived:

$SE(\hat{\theta}) = 0.02476$, $SE(\hat{\alpha}) = 0.02532$ and $SE(\hat{\beta}) = 0.046797$. 95% confidence intervals for the three parameters based on log-likelihood are provided as $(\theta_L, \theta_U) = (0.8095\,, 0.9065)$, $(\alpha_L, \alpha_U) = (0.989\,, 0.9989)$ and $(\beta_L, \beta_U) = (0.01328\,, 0.1967)$.

The value of the log-likelihood function computed at the MLEs is $l_{max} = -180.42$ and the corresponding value of the Akaike Information Criterion ($AIC = 2(K - l_{max}$, with k = 3 being the number of parameters) is 354.84.

We computed the theoretical absolute joint frequencies, by using the PMF in (6) with the MLEs of the parameters $(\theta\,, \alpha\,, \beta)$ they are displayed between brackets in Table 6. Then we aggregated cells in order to obtain for each grouping an aggregate frequency larger than or equal 5; we computed the chi-square statistic $\chi^2 = \sum_{g=1}^{G} \left(\hat{n}_g - n_g\right)^2 / \hat{n}_g$ where $n_g$is the observed count for grouping g, $\hat{n}_g$is its theoretical analog, G is the number of groupings (in this case G = 36). Under the null hypothesis that the bivariate sample comes from the proposed distribution, χ^2 is approximately distributed as a chi-square r. v. with 36 - 3 - 1 = 32 degrees of freedom. The empirical value of $\chi^2$is 18.123; its p-value is 0.994 and being far larger than 0 it refers to a satisfactory fit of the model to the data.

Plugging in the MLEs of the three parameters into (16), one derives the MLE of R, as $\hat{R} = 0.4153$, which represents the estimated probability that the number of aborts in the second period is not smaller than the number of aborts in the first one. By the way, the MLE of R is very close to the standard non-parametric estimate $\tilde{R} = \frac{1}{n}\sum_{i=1}^{n} 1_{x_i \le y_i} = \frac{134}{1380} = 0.353$.

6. Conclusion

In this paper a bivariate discrete Lindley distribution has been derived from a discrete Lindley distribution using Farlie-Gumbel-Morgenstern copula which can be used in many situations where discrete data appear, such as industrial quality control, insurance, health, economics, and marketing, etc.

Some properties of this distribution such as probability generating function, conditional distributions, Pearson's correlation, reliability parameter are studied.

To estimate the parameters of the distribution, three methods of estimation were presented which are ML, MoM, TSML. Then those methods were compared using numerical simulation and have been concluded that the best estimation method for this distribution is ML since it gives the lowest values for Bias, L.CI and MSE. These estimators are characterized by consistency, which means Bias and MSE decrease with increasing sample size but despite that the method of ML requires more time and effort during the estimation process than the two methods MoM and TSML. Where ML estimates the three parameters sat the same time and all together since the other methods estimate the parameters θ, α as a first stage then estimate the parameter β at a later stage. That makes the properties of the parameter β not good compared to the properties of the rest of the parameters.

Finally, a practical application was made on real data to show the appropriateness of the proposed distribution on those data where the results of the estimation showed that the bivariate discrete Lindley distribution has a goodness of fit on those data.

We hope that the proposed model will be a valuable alternative to the existing models dealing with this kind of data set considered here.

Abbreviations

PMF	Probability Mass Function
CDF	Cumulative Distribution Function
FGM	Farlie-Gumbel-Morgenstern
BDL	Bivariate Discrete Lindley
MoM	Method of Moments
MLEs	Maximum Likelihood Estimates
ML	Maximum Likelihood
TSML	Two-Step ML
IFM	Inference Functions for Margins
L.CL	length of Confidence Interval
AIC	Akaike Information Criterion
G	the number of groupings

Declarations

Availability of data and materials

The Data has been collected from the following trusted soccer's site,

https://www.soccer24.com/italy/serie-a-2019-2020/results/

Competing interests

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Authors' contributions

The second author (Rania Shalabi) conceived and designed the paper and wrote the paper. The corresponding author (Yasser Amer) collected the data, provided the analysis tools, and performed the analysis. The third author reviewed the paper and made the corrections. All authors contributed in the part of the distribution's properties, the estimation of the parameters, the simulation study and the practical application.

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