# Exponentiated (Lehmann Type-II) Nadarajah-Haghighi Distribution: Properties and Applications

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# Exponentiated (Lehmann Type-II) Nadarajah-Haghighi Distribution: Properties and Applications

**Dr. Mahmoud Ali Selim and Dr. Nader Fathi**

## Abstract

In this paper we introduce a new distribution called exponentiated (Lehmann type-II) Nadarajah-Haghighi (ENH-II). This version is considered a generalization of the original distribution and is more flexible than it. The statistical distributional properties of the new model are explored, including the reliability function, hazard function, a reverse hazard function, quantile function, and moment generating function. Additionally, the paper derives the moments, incomplete moments, mean deviations, Bonferroni and Lorenz curves, and some of order statistics densities functions. The study employs the maximum likelihood estimation method to estimate the model parameters. Finally, the applicability of the proposed distribution has been demonstrated by application to real data. The application showed the flexibility of the proposed distribution and its superiority over similar distributions,

**Keywords:** Nadarajah-Haghighi distribution; Weibull distribution;; moment; hazard rate function; order statistics; Bonferroni and Lorenz curves; maximum likelihood estimation.

## 1. Introduction

The Nadarajah-Haghighi (N-H) distribution was proposed by Nadarajah and Haghighi (2011) as an alternative of several other common distributions, including, Weibull, exponential, *exponentiated exponential* distributions. The cumulative density function (cdf) and probability density function (pdf) of NH distribution, respectively, are:

$$G(x) = 1 - exp\{1 - (1 + \lambda x)^{\theta}\}, \quad \lambda, \theta > 0, \; x > 0 \tag{1.1}$$

and

$$g(x) = \lambda\theta(1 + \lambda x)^{\theta-1} exp\left\{1 - (1 + \lambda x)^{\theta}\right\}, \; \lambda, \theta > 0, \; x > 0 \tag{1.2}$$

where θ, λ are shape and scale parameters, respectively. The exponential distribution is a special case of (1.1) when $\theta = 1$. Therefore, it can be considered as an extension of exponential distribution. This model has desirable properties as it has the zero mode and sorts of monotone increasing, monotone decreasing and constant of hazard rate functions.

Several generalizations of the Nadarajah-Haghighi (NH) distribution have been proposed in the literature. Some of these include: the exponentiated (Lehmann type 1) Nadarajah Haghighi (ENH) by (Lemonte 2013) , transmuted Nadarajah-Haghighi by (Ahmed et al. 2015) , Kumaraswamy Nadarajah-Haghighi by (Lima 2015), Exponentiated Generalized Nadarajah-Haghighi Distribution by (VedoVatto et al. 2016), the odd Lindley Nadarajah-Haghighi distribution by (Yousof et al. 2017), beta Nadarajah-Haghighi (BNH) distribution by (Cícero et al. 2018), Extended Exponentiated family of distributions by (Alizadeh et al. 2018), Odd Nadarajah-Haghighi Distribution by (Nascimento et al. 2019), beta exponentiated Nadarajah-Haghighi (BENH) distribution by (Saboor et al. 2019), the nadarajah-haghighi Lindley distribution by (Peña-Ramírez et al. 2019), logistic Nadarajah–Haghighi distribution by (Peña-Ramírez et al. 2020), sin Nadarajah-Haghighi (SNH) distribution(Almetwally and Meraou 2022), Topp-Leone Nadarajah-Haghighi distribution by (Yousof and Korkmaz 2017), Lindley exponentiated Nadarajah Haghighi distribution by(Shehata and Yousof 2022), Nadarajah–Haghighi Lomax distribution(Nagarjuna et al. 2022), Type I Half-logistic Nadarajah-Haghighi distribution by (Shrahili et al. 2023), Skewed Nadarajah–Haghighi Distribution by (Chesneau et al. 2020), Burr X exponentiated Nadarajah Haghighi Distribution by (Abdelkhalek 2023). These generalizations of the Nadarajah Haghighi distribution have been used in various applications, such as reliability analysis, survival analysis, and failure time modeling. Gupta et al. (1998), introduced the exponentiated method. Which is a powerful technique for generating new families of distributions from existing ones. By applying the exponentiated method to existing distributions, researchers can develop new models that have more flexibility than the base distribution and can better fit broadly real data sets. From cdf G(x) of an arbitrary parent continuous distribution, the cdf F(x) of Lehmann Type-II class of distributions is defined by:

$$F(x) = 1 - \left(1 - G(x)\right)^{B}, \qquad x > 0 \tag{1.3}$$

with corresponding CDF as

$$f(x) = Bg(x)\left(1 - G(x)\right)^{B-1}, \qquad x > 0 \tag{1.4}$$

Where $\beta > 0$ is a shape parameter, $g(x)$ is the pdf of the parent distribution. Several of the generalized distributions from (1.3) were studied in the literature including, the generalized inverse Weibull distribution by De Gusmao et al. (2011) and the generalized inverse generalized Weibull distribution by Jain et al. (2014).

This paper aims to introduce a new flexible generalization of the Nadarajah-Haghighi distribution specified exponentiated (or Lehmann type-II) Nadarajah-Haghighi (*ENH-II*) distribution and discusses its statistical properties and applications. This paper is arranged as follow. Section 2, defines the exponentiated Nadarajah-Haghighi (ENH-II) distribution. Section 3, discusses the statistical properties of the proposed model. In section 4, the reliability functions are derived. In section 5, the pdf of order statistics of the *ENH*-II distribution is introduced. Section 6, discusses the estimation parameters model by the maximum likelihood method. Section 7, A real data set is utilized to explain the benefit and applicability of the ENH-II distribution. In section 7, the concluding comments are given.

## 2. Exponentiated (Lehmann type-II) Nadarajah-Haghighi Distribution

From (1.3) and (1.1) the CDF of *ENH-II* distribution can be shown as;

Nadarajah-Haghighi **distribution** is a special case of (6) when $\beta = 1$. Some **possible shapes of** pdf and cdf are drawn in Fig. 1, 2 respectively.

$$F(x) = 1 - exp\left\{\beta\left[1 - (1 + \lambda x\ \ )^{\theta}\right]\right\}, \qquad \lambda, \theta, \beta > 0,\ x > 0 \tag{6}$$

and the corresponding PDF as

$$f(x) = \lambda\theta\beta(1 + \lambda x\ \ )^{\theta-1} exp\left\{\beta\left[1 - (1 + \lambda x\ \ )^{\theta}\right]\right\} \quad \lambda, \theta, \beta > 0,\ x > 0 \tag{7}$$

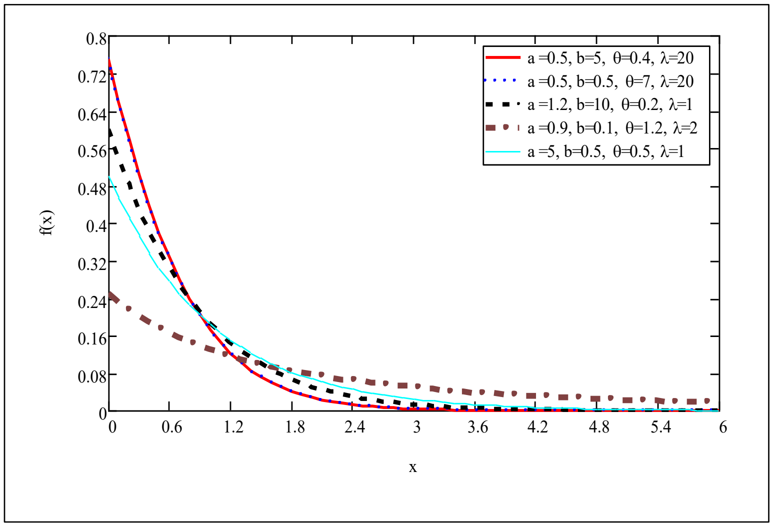


**Fig. 1: Some shapes of *ENH-II* pdf**

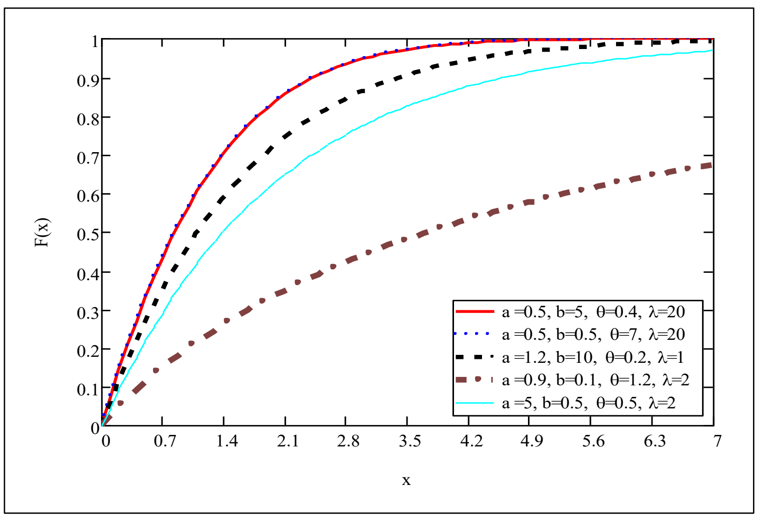


**Fig. 2: Some shapes of *ENH-II* cdf**

The special sub-models of the ENH-II distribution are; Nadarajah-Haghighi $(NH)$ distribution when $\beta = 1$, and the exponential (E) distribution in two ways, either when $\lambda = \theta = 1$ or when $\beta = \theta = 1$.

# 3. Mathematical Properties of *ENH-II* Distribution

This section studies some of the main properties of the ENH-II distribution as follows:

## 3.1 Quantile function

The quantile function Q(q), say $Q(q) = F^{-1}(q)$, of the *ENH-II* distribution can be define as:

$$Q(q) = \lambda^{-1}\left\{\left[1 - \frac{ln(1-q)}{\beta}\right]^{1/\theta} - 1\right\} \qquad 0 \le q \le 1 \;\;,\;\; \beta \neq 0, \theta \neq 0, \lambda \neq 0 \qquad (8)$$

Therefore, the median $M(x)$ of *ENH-II* distribution can be obtained, by setting $q = 0.5$, in Eq (8) as follows:

$$M(x) = \lambda^{-1}\left\{\left[1 - \frac{ln(0.5)}{\beta}\right]^{1/\theta} - 1\right\} \qquad (9)$$

## 3.2 Skewness and kurtosis

The Bowley's skewness measure based on quartiles from Eq (8) ((Kenney and Keeping 1962)) is given by

$$Sk = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)} \qquad (10)$$

and the Moors' kurtosis measure (Moors (1988)) is given by:

$$Ku = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \qquad (11)$$

## 3.3 Random variables generation

The random variables of *ENH-II* distribution can be directly generated based on quantile function in Eq (8) as follows;

$$X = \lambda^{-1}\left\{\left[1 - \frac{ln(1-u)}{\beta}\right]^{1/\theta} - 1\right\} \qquad (12)$$

where $0 \le u \le 1$ is generated number from the uniform distribution $(0, 1)$.

## `3.4 The Moments

If $X\sim$ *ENH-II* distribution, the r-th moment of $X$ denoted as, $\mu_r^{'}$, is:

$$\mu_r^{'} = \beta e^{\beta}\lambda^{-r}\sum_{j=0}^{r}\frac{(-1)^{j+r}}{\beta^{\frac{j}{\theta}+1}}\binom{r}{j}\Gamma\left(\frac{j}{\theta}+1,\beta\right) \quad (13)$$

**Proof**. If X has the pdf in Eq(7), then rth moment is obtained as follows:

$$\mu_r^{'} = E(X^r) = \int_0^{\infty} x^r f(x)dx \quad (14)$$

$$= \theta\lambda\beta\int_0^{\infty} x^r(1+\lambda x)^{\theta-1}e^{\beta(1-(1+\lambda x)^{\theta})}dx, \quad (15)$$

By putting $u^{\frac{1}{\theta}} = 1+\lambda x$, we get

$$\mu_r^{'} = \beta e^{B}\lambda^{-\frac{r}{\alpha}}\int_1^{\infty}\left(u^{\frac{1}{\theta}}-1\right)^{\frac{r}{\alpha}}e^{-\beta u}du. \quad (16)$$

By applying the binomial expansion, then Eq (16) become

$$\mu_r^{'} = \beta e^{\beta}\lambda^{-r}\sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j}\int_1^{\infty}u^{\frac{j}{\theta}}e^{-\beta u}du. \quad (17)$$

By knowing that $\int_1^{\infty}u^{\frac{j}{\theta}}e^{-\beta u}du = \Gamma\left(\frac{j}{\theta}+1,\beta\right)/\beta^{\frac{j}{\theta}+1}$, we get the *r-th* moment of *ENH-II* as follows:

$$\mu_r^{'} = \beta e^{\beta}\lambda^{-r}\sum_{j=0}^{r}\frac{(-1)^{j+r}}{\beta^{\frac{j}{\theta}+1}}\binom{r}{j}\Gamma\left(\frac{j}{\theta}+1,\beta\right)$$

If $\beta = 1$, we can obtain the moments of Nadarajah and Haghighi distribution, as special case, as follows:

$$\mu_r^{'} = \lambda^{-r}e\sum_{j=0}^{r}(-1)^{r+j}\binom{r}{j}\Gamma\left(\frac{j}{\theta}+1,1\right)$$

this result agrees with Nadarajah and Haghighi (2011)

The first two r-th moments can be obtained from (13) as follows:

$$\mu_1^{'} = E(X) = \beta e^{\beta}\lambda^{-1}\sum_{j=0}^{1}\frac{(-1)^{j+1}}{\beta^{\frac{j}{\theta}+1}}\binom{1}{j}\Gamma\left(\frac{j}{\theta}+1,\beta\right), \quad (18)$$

$$\mu_2^{'} = E(X^2) = \beta e^{\beta}\lambda^{-2}\sum_{j=0}^{2}\frac{(-1)^{j+2}}{\beta^{\frac{j}{\theta}+1}}\binom{2}{j}\Gamma\left(\frac{j}{\theta}+1,\beta\right) \quad (19)$$

and variance can be calculated as follows

$$Var(X) = \mu_2^{'} - [\mu_1^{'}]^2 \quad (20)$$

Based on the central moments $\mu_r$ and non-central moments $\mu_r^{'}$ we can calculate the central moments $\mu_r$ and the cumulants $k_r$. By using $\mu_r^{'}$ in (13), we get $\mu_r = \sum_{k=0}^{r}(-1)^k\binom{r}{k}\mu_1^{'k}\mu_{r-k}^{'}$ and $k_r = \mu_r^{'} - \sum_{k=1}^{r-1}\binom{r-1}{k-1}k_r\mu_{r-k}^{'}$. Therefore, we can get the skewness and kurtosis $\gamma_1 = k_3/k_2^{3/2}$ and $\gamma_2 = k_4/k_2^2$ respectively.

# 3.5 The moment generating function

The moment generating function of $X \sim$ *ENH-II* distribution is

$$M_x(t) = \beta e^{\beta} \lambda^{-\frac{r}{\alpha}} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{j+\frac{r}{\alpha}} \mathrm{t}^{\mathrm{r}}}{\beta^{\frac{j}{\theta}+1} \mathrm{r}!} \binom{r/\alpha}{j} \Gamma\left(\frac{j}{\theta}+1, \beta\right) \quad (21)$$

**Proof**. Starting from the mathematical definition of moment generating function (mgf) which is

$$M_x(t) = \int_0^{\infty} e^{tx} \ f(x)dx$$

and by knowing that, $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$, we get

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x)dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' \quad (22)$$

and by inserting (16) in to (22) produces the mgf of ENH-II distribution as in (21).

### 3.6 Incomplete moments

by putting (7) in (23), we obtain

The s-th incomplete moment of ENH-II distribution is

$$m_r(s) = \beta e^{\beta} \lambda^{-r} \sum_{j=0}^{r} \frac{(-1)^{j+r}}{\beta^{\frac{j}{\theta}+1}} \binom{r}{j} \Gamma\left(\frac{j}{\theta}+1, \beta(1+\lambda s \ \ )^{\theta}\right) \quad (24)$$

**Proof**. The definition of the s-th incomplete moment of $X$, is

$$m_r(s) = E(X^r | X > s) = \int_s^{\infty} x^r f(x)dx \quad (23)$$

by inserting (7) in (23), we get

$$m_r(s) = \beta\lambda\theta \int_s^{\infty} x^r (1+\lambda x)^{\theta-1} e^{\beta(1-(1+\lambda x)^{\theta})} dx \quad (25)$$

and by replacing $u = (1+\lambda x \ \ )^{\theta}$, we get

$$m_r(s) = \beta e^{\beta} \lambda^{-r} \int_{(1+\lambda s)^{\theta}}^{\infty} \left(u^{\frac{1}{\theta}} - 1\right)^r e^{-\beta u} du \quad (26)$$

and by expanding the binomial $\left(u^{\frac{1}{\theta}} - 1\right)^r$, we get

$$m_r(s) = \beta \lambda^{-r} e^{\beta} \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} \int_{(1+\lambda s)^{\theta}}^{\infty} u^{\frac{j}{\theta}} e^{-\beta u} du \quad (27)$$

By knowing that, $\int_{(1+\lambda s)^{\theta}}^{\infty} u^{\frac{j}{\theta}} e^{-\beta u} du = \Gamma\left(\frac{j}{\theta}+1, \beta(1+\lambda s)^{\theta}\right) / \beta^{\frac{j}{\theta}+1}$, we obtain the r-*th* upper incomplete moment of *ENH-II* distribution as in (24).

The first incomplete moment of the ENH-II distribution is got by placing r=1 in (24), as follows:

$$m_1(s) = \beta e^{\beta} \lambda^{-1} \sum_{j=0}^{1} \frac{(-1)^{j+1}}{\beta^{\frac{j}{\theta}+1}} \binom{1}{j} \Gamma\left(\frac{j}{\theta}+1, \beta(1+\lambda s)^{\theta}\right). \quad (28)$$

## 3.7 Mean deviations

The mean deviations about the median ($\delta_1(X)$) of $X$ is

$$\delta_1(X) = E(|X - M|) = \int_0^\infty |X - M| f(x)dx$$
$$= 2MF(M) - M - \mu_1' + 2\int_M^\infty xf(x)dx$$
$$= \mu_1' - 2H_1(M) \qquad (29)$$

$where\ \mu_1' = E(X)$ and $M =$ median(X)

and the mean deviations about the mean ($\delta_2(X)$) of $X$ is

$$\delta_2(X) = E(|X - \mu_1'|) = \int_0^\infty |X - \mu_1'| f(x)dx$$
$$= 2\mu_1' F(\mu_1') - 2\mu_1' + 2\int_{\mu_1'}^\infty xf(x)dx$$
$$= 2\mu_1' F(\mu_1') - 2H_1(\mu_1') \qquad (30)$$

where $F(\mu_1')$ is obtained from (6), and $H_1(s)$ is the s-th lower incomplete moment of the ENH-II distribution as follows:

$$H_1(s) = \int_0^{\mu_1'} xf(x)dx$$
$$= \beta e^{\beta} \lambda^{-1} \sum_{j=0}^{1} \frac{(-1)^{j+1}}{\beta^{\frac{j}{\theta}+1}} \binom{r}{j} \left[ \Gamma\left(\frac{j}{\theta} + 1, \beta\right) - \Gamma\left(\frac{j}{\theta} + 1, \beta(1 + \lambda s\ \ )^{\theta}\right) \right] \qquad (31)$$

## 3.8 Bonferroni and Lorenz curves

The Bonferroni curve (see (Bonferroni (1930)), is

$$B(p) = \frac{1}{p\mu_1'} \int_0^q xf(x)dx = \frac{\mu_1' - m_1(q)}{p\mu_1'} \qquad (32)$$

and Lorenz curve is

$$L(p) = \frac{1}{\mu_1'} \int_0^q xf(x)dx = \frac{\mu_1' - m_1(q)}{\mu_1'} \qquad (33)$$

where $\mu_1' = E\ (X)$ and $q = Q(p)$ is calculated from (8) for a given probability ($p$), and $m_1(q)$ is the first incomplete moment from (28).Then $B(p)$ can be written as:

$$B(p) = \frac{1}{p} - \frac{\beta e^{\beta} \lambda^{-\frac{1}{\alpha}} \sum_{j=0}^{1/\alpha} \frac{(-1)^{j+\frac{1}{\alpha}}}{\beta^{\frac{j}{\theta}+1}} \binom{1/\alpha}{j} \left[ \Gamma\left(\frac{j}{\theta} + 1, \beta(1 + \lambda q^{\alpha})^{\theta}\right) \right]}{p\mu_1'} \qquad (34)$$

and

$$L(p) = 1 - \frac{\beta e^{\beta} \lambda^{-\frac{1}{\alpha}} \sum_{j=0}^{1/\alpha} \frac{(-1)^{j+\frac{1}{\alpha}}}{\beta^{\frac{j}{\theta}+1}} \binom{1/\alpha}{j} \left[ \Gamma\left(\frac{j}{\theta} + 1, \beta(1 + \lambda q^{\alpha})^{\theta}\right) \right]}{\mu_1'} \qquad (35)$$

## 4. Reliability Analysis

In this section, the survival $s(t)$, failure rate $h(t)$, reversed hazard $r(t)$ and the cumulative failure rate $H(t)$ functions of *EPGW-II* distribution are derived.

### 4.1 The survival function

The survival function $R(t)$ of the *EPGW-II* distribution can be derived using the cumulative distribution function in (5) as follows

$$R(t) = 1 - F(x) = \left[e^{1-(1+\lambda x^{\alpha})^{\theta}}\right]^{\beta} \ , \ \ t > 0 \qquad (36)$$

### 4.2 The hazard function

For a continuous distribution with pdf $f(x)$ and cdf $F(x)$, the hazard rate function for any time is defined as follows

$$h(t) = \lim_{\Delta t \to 0} \frac{P(T < t + \Delta x | T > t)}{\Delta t} = \frac{f(t)}{1 - F(t)}$$

Subsequently, the hazard rate for any time of the *EPGW-II* distribution can be determined using the cdf and pdf in Eqs. (5), (6) as follow:

$$h(t) = \alpha\beta\lambda\theta t^{\alpha-1}(1 + \lambda t^{\alpha})^{\theta-1}, \ \ t > 0 \qquad (37)$$

The plot of the hazard function of *ENH-II* distribution for selected values of parameters are showed in Fig.3.

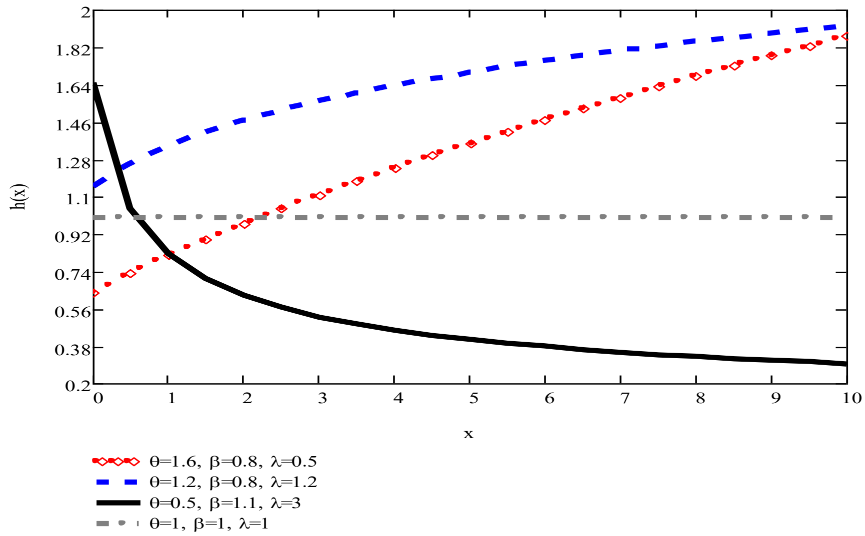


**Fig. 3: Some shapes of the** *ENH-II* hazard rate **function**

## 4.٣ The reversed hazard and cumulative hazard rate functions

The reversed hazard $r(t)$ and the cumulative hazard rate $H(t)$ functions of EPGW-II distribution are given, respectively, as follow

$$r(t) = \frac{\alpha\beta\lambda\theta x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1}\left[e^{1-(1+\lambda x^{\alpha})^{\theta}}\right]^{\beta}}{1-\left[e^{1-(1+\lambda x^{\alpha})^{\theta}}\right]^{\beta}}, \quad t > 0, \tag{38}$$

and

$$H(t) = -\ln\left[1 - exp\left\{\beta\left[1-(1+\lambda x^{\alpha})^{\theta}\right]\right\}\right] \ , \quad t > 0. \tag{39}$$

## 5. Order Statistics

The order statistics arise naturally in many areas of statistical theory and practice which makes it one of the important statistical topics. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics of a random sample drawn from a continuous distribution with cdf $F(x)$ and pdf $f(x)$, then the pdf of $X_{(k)}$ is given by:

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}, \quad k = 1,2,\dots,n \tag{40}$$

Let $X$ is a random variable of *EPGW-II* distribution, then by substituting (6) and (7) into equation (40), we get the $kth$ order statistics of *EPGW-II* density function as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!\,(n-k)!}\alpha\lambda\theta\beta x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1}\left[1-\left(e^{1-(1+\lambda x^{\alpha})^{\theta}}\right)^{\beta}\right]^{k-1} \times \left[e^{1-(1+\lambda x^{\alpha})^{\theta}}\right]^{\beta(n-k+1)} \quad (41)$$

The pdf of order statistics when $k = 1$ and when $k =$ n are

$$f_{1:n}(x) = n\alpha\beta\lambda\theta x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1}\left[e^{1-(1+\lambda x^{\alpha})^{\theta}}\right]^{n\beta} \quad (42)$$

and

$$f_{n:n}(x) = n\alpha\beta\lambda\theta x^{\alpha-1}(1+\lambda x^{\alpha})^{\theta-1}\left[1-\left(e^{1-(1+\lambda x^{\alpha})^{\theta}}\right)^{\beta}\right]^{n-1}\left[e^{1-(1+\lambda x^{\alpha})^{\theta}}\right]^{\beta} \quad (43)$$

respectively.

# 6. Maximum Likelihood Estimation

Let $x_1, x_1, \dots, x_n$ is a random sample of the *ENH-II* distribution. The likelihood function (LF) is

$$L(\lambda\theta\beta|x) = (\lambda\theta\beta)^n \prod_{i=1}^{n}\left(\lambda x_i \;\; +1\right)^{\theta-1}\left(e^{1-\left(\lambda x_i \;\; +1\right)^{\theta}}\right)^{\beta} \quad (44)$$

and the expression of the log-likelihood function is given by

$$logL = n\,log(\lambda\theta\beta) + (\theta-1)\sum_{i=1}^{n} ln\left(\lambda x_i \;\; +1\right) + \beta - \beta\sum_{i=1}^{n}\left(\lambda x_i \;\; +1\right)^{\theta} \quad (45)$$

By taking the partial derivatives of the above equation we get

$$\frac{\partial\, ln\, L}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\left(\lambda x_i \;\; +1\right)^{\theta} + 1\,, \quad (46)$$

$$\frac{\partial\, ln\, L}{\partial\lambda} = \frac{n}{\lambda} - \beta\theta\sum_{i=1}^{n} x_i\;\left(\lambda x_i \;\; +1\right)^{\theta-1} + (\theta-1)\sum_{i=1}^{n}\frac{x_i}{\lambda x_i \;\; +1}\,, \quad (48)$$

$$\frac{\partial\, ln\, L}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln\left(\lambda x_i \;\; +1\right) - \beta\sum_{i=1}^{n} ln\left(\lambda x_i \;\; +1\right)\left(\lambda x_i \;\; +1\right)^{\theta}\,. \quad (49)$$

The simultaneous solutions of the nonlinear likelihood equations $\frac{\partial \ln L}{\partial\beta} = \frac{\partial \ln L}{\partial\lambda} = \frac{\partial \ln L}{\partial\theta} = 0$, yields the ML estimates for the unknown model parameters.

The previous equations cannot be analytically solved, but the statistical software can be used to solve them numerically by using iterative techniques like the Newton-Raphson algorithm.

# 7. Real Data Illustration

The survival times (in weeks) of 33 patients suffering from acute myelogenous leukemia (see Feigl and Zelen [[25]]) Table (1).. The descriptive statistics for the survival times are display in Table (1). It is clear that the distribution of data has a heavy right tail.

Table (1) Descriptive Statistics for the survival times data

| Mean | Median | Mode | Variance | Skewness | Kurtosis | Min | Max | N |
|---|---|---|---|---|---|---|---|---|
| 40.879 | 22 | 4 | 2181.172 | 1.16457 | 0.1221 | 1 | 156 | 33 |

Table 2. The estimates and their standard errors (in parentheses) for the survival times data

| Model | Estimates | | | | |
|---|---|---|---|---|---|
| | $\theta$ | $\lambda$ | $\alpha$ | $b$ | $\gamma$ |
| *EW* | - | 0.1680 (0.5769) | 0.5998 (0.6057) | - | 1.5748 (2.9553) |
| *ENH* | 0.3151 (0.2651) | 0.6516 (3.214) | - | - | 1.8757 (3.5269) |
| *NH* | 0.4897 (0.1482) | 0.0998 (0.0701) | - | - | - |
| *EE* | - | 0.6785 (0.1448) | - | - | 0.0188 (0.0048) |
| *W* | - | 0.0628 (0.0298) | 0.7763 (0.1073) | - | - |
| *E* | - | 0.0245 (0.0043) | - | - | - |
| *E2NH* | **0.6891 (0.1740)** | **0.8028 (1.9689)** | | **0.1121 (0.2577)** | |

Table 3. Goodness-of-fit statistics for the survival times data

| Model | Kolmogorov-Smirnov | | $W^*$ | $A^*$ | -L |
|---|---|---|---|---|---|
| | K–S | P-value | | | |
| *EW* | 0.1366 | 0.5692 | 0.0954 | 0.6455 | 153.562 |
| *ENH* | 0.1301 | 0.6315 | 0.1103 | 0.6949 | 153.746 |
| *NH* | 0.1393 | 0.5441 | 0.1002 | 0.6659 | 153.743 |
| *EE* | 0.1384 | 0.5521 | 0.0966 | 0.6691 | 153.652 |
| *W* | 0.1366 | 0.5689 | 0.0948 | 0.6508 | 153.587 |
| *E* | 0.2182 | 0.0864 | 0.0973 | 0.6730 | 155.450 |
| *E2NH* | **0.1302** | **0.6307** | **0.0909** | **0.6159** | **153.22** |

To initially check that the data is suitable for the proposed distribution (see for more details Aarset (1987)), we can use the total time on test (TTT) plot to recognize the shape of hazard rate function graphically. Figure 4 presents the TTT plot of our data, which represents a decreasing hazard rate function. This Figure implies the appropriateness of the ENH-II distribution to fit our datasets.

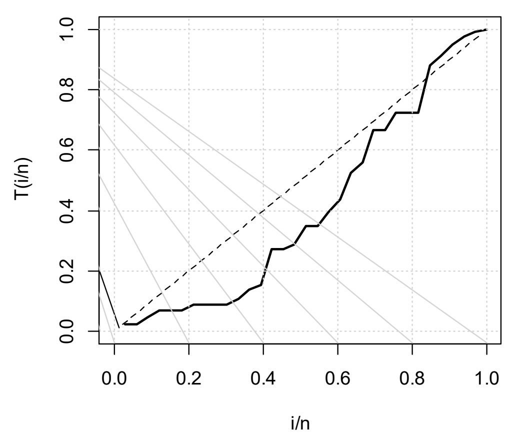


**Fig.4: TTT plot for dataset**

ENH-II distribution has been fitted to the real data and the results has been compared with the results of the other distributions close to it,which are exponentiated Nadarajah-Haghighi (ENH) distribution, Nadarajah-Haghighi (NH) distribution, exponentiated Weibull (EW) distribution, Weibull (W) distribution, exponentiated exponential (EE) distribution and exponential (E) distribution. The numerical computations needed for the analysis are conducted utilizing the R software (Team (2013)). To compare the previous distributions and assess the goodness of fit, we employ various widely recognized Goodness-of-Fit statistics. like, Cramér-von Mises ($W^*$), Anderson Darling ($A^*$), Kolmogorov-Smirnov ($KS$), Maximized Loglikelihood ($-L$). The model with a minimum value of Goodness-of-Fit statistics is the best model to fit the data.

The MLEs and their standard errors of the fitted models are presented in Table 2. As well, the values of Goodness-of-Fit statistics of the fitted models are presented in Table 3. Tables 2 and 3, show that the proposed EPGW-II model gives smallest values for the Goodness-of-Fit statistics. The plots of histogram and estimated pdf for the EPGW-II distribution and other fitted distributions for the survival times data are displayed in Fig. 5 and Fig. 6, respectively. The plots also demonstrate that the EPGW-II distribution offers a better fit than other distributions for the survival times data.

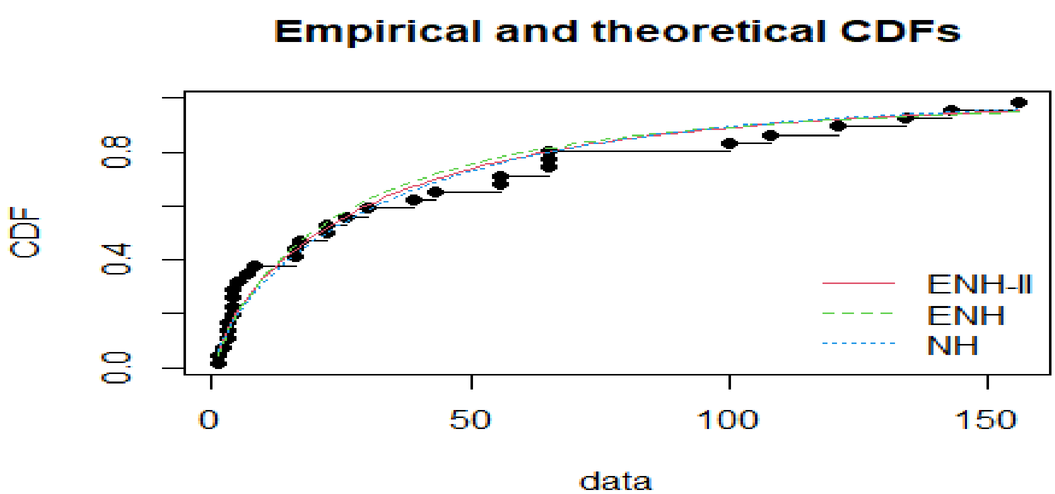


**Fig. 5: Histogram and estimated densities for dataset** for ENH-II, ENH, NH, EE distributionsFig. 6: Empirical and fitted cdfs for our dataset for ENH-II, ENH, NH distributions.

## 8. Conclusion

This paper introduces a novel four-parameter probability distribution known as the exponentiated (Lehmann type-II) power generalized Weibull distribution. This distribution encompasses well-known sub-models such as Weibull, Rayleigh, exponential, power generalized Weibull and Nadarajah-Haghighi distributions. This large number of special cases for the new distribution confirms its importance and wide range of practical applications. The statistical properties of this new model are investigated, including the hazard rate function, quantile function, order statistics, moments, incomplete moments, mean deviations, and Bonferroni and Lorenz curves. The hazard function of the proposed model exhibits a diverse range of shapes, including increasing, decreasing, bathtub, unimodal, and constant shapes. The model parameters are estimated using the maximum likelihood method. Empirical evidence from real-world applications demonstrates that this new distribution is highly beneficial for analyzing lifetime data, outperforming commonly used distributions for fitting this type of data. As future points, we suggest further study on the distribution properties, as well as parameter estimation using both Bayesian and non-Bayesian estimation methods and comparing them.

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# التوزيع الأسي (ليمان النوع الثاني) ندراجا-حقيقي: الخصائص والتطبيقات

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**الملخص**

في هذا البحث نقدم توزيعًا جديدًا يسمى التوزيع الأسي (ليمان النوع الثاني) ندراجى-حقيقي تعتبر هذه النسخة تعميما لتوزيعاته الأصلية وأكثر مرونة منها. تمت ايجاد الخصائص الاحصائية للتوزيع الجديد. تم الحصول على دالة الصلاحية، ودالة الخطر، ودالة الخطر العكسي، والدالة الربعيات، ودالة المولدة للعزوم، ودوال الاحصائيات الترتيبية، علاوة على ذلك، تم اشتقاق العزوم ، العزوم غير الكاملة، متوسط الانحرافات، منحنيات Bonferroni و Lorenz، دوال إحصائيات الترتيبية. تم تقدير معالم النموذج باستخدام طريقة الإمكان الأكبر. تم إثبات قابلية تطبيق التوزيع المقترح من خلال التطبيق على البيانات الحقيقية. وأظهر التطبيق مرونة التوزيع المقترح وتفوقه على التوزيعات المشابهة.

**الكلمات الافتتاحية**: توزيع نادراجا-حقيقي؛ العزوم؛ دالة معدل الخطر؛ الاحصائيات الترتيبية ; منحنيات بونفيروني ولورينز; تقدير الإمكان الاكبر