Approximated Characteristics of Bivariate Discrete Time Series with Missing Data

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Approximated Characteristics of Bivariate Discrete Time Series with Missing Data

Dr. Amira El-Desokey; Dr. Mohamed Alargat and Dr. Mohamed Ghazal

Abstract- The extended finite Fourier transformation is an effective mathematical method for analyzing time series data with vector values. In this study, the transformation is applied to (n + m) time series data, and the approximations obtained are used to create usable features for further analysis. This technique could be useful in the field of climate science, where missing data can be a significant challenge. Researchers may more accurately analyses climate data using the extended finite Fourier transformation, even when some observations are missing at random. This can lead to a better understanding of climate patterns and trends over time, which is necessary for forecasting future changes and developing effective mitigation policies. Overall, the extended finite Fourier transformation is an interesting development in the field of time series analysis, with several possible applications in a variety of fields. We should expect even more spectacular developments in the coming years as academics continue to explore its powers and perfect its methodologies.

Key Words- Discontinuous Time Stable Processes; Tapered Data; Fourier transform, Unobserved Data, Whishart Distribution.

1. Introduction

Bivariate discrete time series with missing data are commonly encountered in various fields such as economics, finance, and engineering. The analysis of such data is essential for understanding the underlying patterns and relationships between the variables. However, missing data can pose a significant challenge in accurately estimating the characteristics of these time series. To address this issue, researchers have developed various methods for approximating the characteristics of discrete bi-variate time series with missing Values. The importance of this research lies in its potential to improve our understanding of complex systems and inform decision-making processes in various domains. This research aims to provide an overview of the most recent approaches for approximating the characteristics of bivariate discrete time series with missing data and their applications in different fields. The importance of Bivariate Discrete Time Series with Missing Data lies in its ability to provide a framework for analyzing and understanding complex data sets that contain missing values. In many real-world scenarios, data is often incomplete due to various reasons such as measurement errors, equipment failures, or human errors. This can lead to biased or inaccurate results if not handled properly. The paper proposes a method for approximating the characteristics of bivariate discrete time series with missing data using a combination of imputation and estimation techniques. This approach allows researchers to make more accurate predictions and draw meaningful conclusions from incomplete data sets. Furthermore, several different sectors can benefit from the proposed approach like finance, economics, social sciences, and engineering. It can help researchers identify patterns and relationships between variables that would otherwise be obscured by missing data. Overall, the importance of this research lies in its potential to improve the accuracy and reliability of data analysis in various fields by providing a robust framework for handling missing data.Several authors, including D. R. Brillinger [1], R. Dahlhaus [3], M. Ghazal and E. Farag [4], and A. El-Desokey [9] who investigated "a few characteristics of discontinuous extended Fourier transformation of missing data," have looked into the problem of predicting the power spectrum, the auto-covariance function, and continual -time processes' spectrum measure, "The spectrum Analysis of firmly fixed continual time process" and "Approximated Characteristics for spectrum Predictions of Second-Order with Missing data" were researched by Ghazal, Mokaddis, and El-Desokey [10,11].

The following is a brief overview of the manuscript: the first section is "Introduction," to our investigation; we create approximated characteristics of estimations for $\underline{\Gamma}$ and $\alpha(\lambda)$, In Section 2. In Section 3, we explored the Approximated characteristics of the Extending Fourier transform with missed data. Our theoretical investigation into the climate is applied in Section 4.

2. Approximated Characteristics Of Estimates required parameters $\underline{\Gamma}$, $\alpha(\lambda)$

Throughout this section, we focus on the difficulty in identifying an m -vector $\underline{\Gamma}$ and a $m \times n$ filter $\{\alpha(\lambda)\}$ in such a way that

$$\underline{\Gamma} + \sum_{u=-\infty}^{\infty} \alpha(t-\lambda)\delta(\lambda) \qquad , \tag{2.1}$$

Which is quite actually closer to $\hbar(t)$. Assume that we quantify proximity using the $m \times m$, Hermitian matrix.

$$E\left\{[\hbar(t) - \Gamma - \sum_{\lambda=-\infty}^{\infty} \alpha(t-\lambda)\delta(\lambda)][\hbar(t) - \underline{\Gamma} - \sum_{\lambda=-\infty}^{\infty} \alpha(t-\lambda)\delta(\lambda)]^T\right\}, \tag{2.2}$$

Theorem 2.1

Assume $(n+m)$ a steady second-order process with vector values formatted as

$$\varpi(t) = [\delta(t) \quad \hbar(t)]^T, t = 0, \pm 1, \pm 2, ... \tag{2.3}$$

With $\delta(t)$, n values and $\hbar(t)$, m values.

Using mean

$$E\delta(t) = \vartheta_\delta \ , \ E\hbar(t) = \vartheta_\hbar \ , \tag{2.4}$$

and auto-covariance functions

$$\begin{aligned} &E\{[\delta(t+\lambda) - \vartheta_x][\delta(t) - \vartheta_x]^T\} = \vartheta_{\delta\delta}(\lambda), \\ &E\{[\delta(t+\lambda) - \vartheta_\delta][\hbar(t) - \vartheta_\hbar]^T\} = \vartheta_{\delta\hbar}(\lambda), \\ &E\{[\hbar(t+\lambda) - \vartheta_\hbar][\hbar(t) - \vartheta_\hbar]^T\} = \vartheta_{\hbar\hbar}(\lambda) \quad , \end{aligned} \tag{2.5}$$

Consider that $\vartheta_{\delta\delta}(\lambda)$, $\vartheta_{\hbar\hbar}(\lambda)$ are completely summable, that $f_{\delta\delta}(g)$, $f_{\delta\hbar}(g)$ and $f_{\hbar\delta}(g)$ are provided by

$$f_{\delta\delta}(g) = (2\pi)^{-1} \sum_{\lambda=-\infty}^{\infty} \vartheta_{\delta\delta}(\lambda)\, Exp(-ig\lambda) \quad ,$$

$$f_{\delta\hbar}(g) = (2\pi)^{-1} \sum_{\lambda=-\infty}^{\infty} \vartheta_{\delta\hbar}(\lambda)\, Exp(-ig\lambda) \quad ,$$

(2.6)

$$f_{\hbar\hbar}(g) = (2\pi)^{-1} \sum_{\lambda=-\infty}^{\infty} \vartheta_{\hbar\hbar}(\lambda)\, Exp(-ig\lambda) \ ,$$

$for\ -\infty < g < \infty$.

Thereby $f_{\delta\delta}(g)$ is a matrix that is nonsingular, $-\infty < g < \infty$. Hence, the, $\underline{\Gamma}$, and $\alpha(\lambda)$ minimizing (2.2) as,

$$\underline{\Gamma} = \vartheta_{\hbar} - \left(\sum_{\lambda=-\infty}^{\infty} \alpha(\lambda) \right) \vartheta_{\delta} = \vartheta_{\hbar} - J(0)\vartheta_{\delta} \ , \tag{2.7}$$

and

$$\alpha(\lambda) = (2\pi)^{-1} \int_0^{2\pi} J(\tau)\, Exp\{i\lambda\tau\}\, d\tau \ , \tag{2.8}$$

where

$$J(g) = f_{\hbar\delta}(g) f_{\delta\delta}(g)^{-1} \ , \tag{2.9}$$

Summability of the $\{\alpha(\lambda)\}$-filter. The Minimal accomplishment is

$$\int_0^{2\pi} [\, f_{\hbar\hbar}(\tau) - f_{\hbar\delta}(\tau) f_{\delta\delta}(\tau)^{-1} f_{\delta\hbar}(\tau)] d\tau. \tag{2.10}$$

In this case, the transferring function of $m \times n$ filtering that provides a required minimal is denoted by $J(g)$, which is the regression coefficient of a complex case of $\hbar(t)$ on $\delta(t)$ at frequency g.

Proof.

Assume $J(g)$ to be the transfer function of $\alpha(\lambda)$ and define it as (2.8). So we can show,

$$E\left\{[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)][\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]^{T}\right\}$$

$$=Cov[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]+E[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]\times$$

$$\times E[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]^{T}$$

$$=E\left\{\left([\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]-E[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]\right)\times\right.$$

$$\left.\times\left([\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]^{T}-E[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]\right)^{T}\right\}+$$

$$+E[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]E[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]^{T}$$

$$=\int_{-\pi}^{\pi}[f_{\hbar\hbar}(\tau)-f_{\hbar\delta}(\tau)f_{\delta\delta}^{-1}(\tau)f_{\delta\hbar}(\tau)]d\tau+\int_{-\pi}^{\pi}[J(\tau)f_{\delta\delta}(\tau)-f_{\hbar\delta}(\tau)]f_{\delta\delta}^{-1}(\tau)\times$$

$$\times[J(\tau)f_{\delta\delta}(\tau)-f_{\hbar\delta}(\tau)]^{T}d\tau+[\vartheta_{\hbar}-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\vartheta_{\delta}][\vartheta_{\hbar}-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\vartheta_{\delta}]^{T}$$

$$E\left\{[\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)][\hbar(t)-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\delta(\lambda)]^{T}\right\}\geq$$

$$\geq\int_{-\pi}^{\pi}[f_{\hbar\hbar}(\tau)-f_{\hbar\delta}(\tau)f_{\delta\delta}^{-1}(\tau)f_{\delta\hbar}(\tau)]d\tau$$

Suppose

$$\vartheta_{\hbar}-\underline{\Gamma}-\sum_{\lambda=-\infty}^{\infty}\alpha(t-\lambda)\vartheta_{\delta}=0\quad,\qquad(2.10)$$

so

$$\underline{\Gamma} = \vartheta_{\hbar} - \sum_{\lambda=-\infty}^{\infty} \alpha(t-\lambda)\vartheta_{\delta} = \vartheta_{\hbar} - J(0)\vartheta_{\delta} \quad ,$$

and

$$J(\tau) f_{\delta\delta}(\tau) - f_{\hbar\delta}(\tau) = 0$$

$$\Rightarrow \qquad J(\tau) = f_{\hbar\delta}(\tau)\, f_{\delta\delta}(\tau)^{-1} \quad ,$$

With (2.7) and (2.8), the Minimal accomplishment score is defined by

$$\int_0^{2\pi} [f_{\hbar\hbar}(\tau) - f_{\hbar\delta}(\tau) f_{\delta\delta}^{-1}(\tau) f_{\delta\hbar}(\tau)] d\tau \, .$$

3- The Approximated Characteristics Of The Extending Fourier Transform With Missed Data.

Let $\partial_a^{(T)}(g)$ represents the definition of the discontinuous extended finite Fourier transformation:

$$\partial_a^{(T)}(g) = \left[2\pi \sum_{t=0}^{T-1} \left(\Re_a^{(T)}(t) \right)^2 \right]^{-1/2} \sum_{t=0}^{T-1} \Re_a^{(T)}(t) \; z_a(t) \exp\{-igt\}, \quad -\infty < g < \infty$$

(3.1)

Such that

$$z_a(t) = \eta_a(t) \varpi_a(t) \quad , \quad a = 1,2,..,\min(n,m) \quad , \qquad (3.2)$$

Probabilistic stable observations are denoted by $\delta_a(t), \hbar_a(t)$, and a sequence of random variables that fit Bernoulli distribution, $\eta_a(t)$, is a stochastically independent of $\delta_a(t), \hbar_a(t)$, and meets the condition that,

$$\eta_a(t) = \begin{cases} 1 & , \text{ if } \delta_a(t), \hbar_a(t) \text{ are existed;} \\ 0 & , \; otherwise. \end{cases}$$

(3.3)

Let $\eta_a(t)$ be a randomly distributed variable that is independent and identical.

$$P[\eta_a(t) = 1] = p_a \quad ,$$
$$P[\eta_a(t) = 0] = q_a \quad ,$$

(3.4)

Where $p_a + q_a = 1$. For any t outside the interval $[0,T]$, the data window functional $\Re_a^{(T)}(t) = \Re_a^{(T)}(t/T), t \in (0,T)$ has limited variation and disappears.

Assumption

If $\Re_a^{(T)}(t), t \in R$, $a = \overline{1,r}$ has no non-vanishing values and its variation is limited, then,

$$A_{a_1,\ldots a_k}(g) = \sum_{t=0}^{T-1} \left[\prod_{j=1}^{l} \Re_{a_j}^{(T)}(t) \right] \exp\{-igt\} \quad ,$$

for $-\infty < g < \infty$ and $a_1, \ldots, a_k = 1,2,\ldots,n$. According to the following theorem, we may obtain an approximation to the characteristics of $z_a(t)$, which are expressed as (3.2).

Theorem3.1

We assume that $z_a(t) = \eta_a(t)\varpi_a(t)$, $a = 1,2,\ldots,\min(n,m)$ is a missing data set for a stable stochastic process $\delta_a(t), \hbar_a(t)$, $a = 1,2,\ldots,\min(n,m)$, and that $\eta_a(t)$ is a series of stochastic process following the Bernoulli distribution, variables satisfying (3.1) and (3.4). Then,

$$E\{z_a(t)\} = 0 \quad , \tag{3.5}$$

$$Cov\{z_{a_1}(t_1), z_{a_2}(t_2)\} = p_{a_1a_2} \begin{bmatrix} \vartheta_{\delta\delta}(\lambda) & \vartheta_{\delta\hbar}(\lambda) \\ \vartheta_{\hbar\delta}(\lambda) & J(\tau)\vartheta_{\delta\delta}(\lambda)J(\tau)^T \end{bmatrix}, \tag{3.6}$$

$$Cov\{z_{a_1}(t_1), z_{a_2}(t_2)\} =$$
$$= p_{a_1a_2} \begin{bmatrix} \int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon & \int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\vartheta J(\tau)^T \\ J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon & J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon J(\tau)^T \end{bmatrix}, \tag{3.7}$$

Proof

Considering that $\eta_a(t)$ is independent and $X(t)$ is a strongly stable series, we obtain (3.5).

$$Cov\left\{z_{a_1}(t_1), z_{a_2}(t_2)\right\}=$$

$$= Cov\left\{\eta_{a_1}(t)\varpi_{a_1}(t), \eta_{a_2}(t)\varpi_{a_2}(t)\right\}$$

$$= Cov\left\{\begin{bmatrix}\eta_{a_1}(t_1)\delta_{a_1}(t_1)\\ \eta_{a_1}(t_1)\hbar_{a_1}(t_1)\end{bmatrix}, \begin{bmatrix}\eta_{a_2}(t_2)\delta_{a_2}(t_2)\\ \eta_{a_2}(t_2)\hbar_{a_2}(t_2)\end{bmatrix}^T\right\}$$

$$= E\begin{bmatrix}\eta_{a_1}(t_1)\delta_{a_1}(t_1)\eta_{a_2}(t_2)\delta_{a_2}(t_2) & \eta_{a_1}(t_1)\delta_{a_1}(t_1)\eta_{a_2}(t_2)\hbar_{a_2}(t_2)\\ \eta_{a_1}(t_1)\hbar_{a_1}(t_1)\eta_{a_2}(t_2)\delta_{a_2}(t_2) & \eta_{a_1}(t_1)\hbar_{a_1}(t_1)\eta_{a_2}(t_2)\hbar_{a_2}(t_2)\end{bmatrix}.$$

$$= \begin{Bmatrix} E\left[\eta_{a_1}(t_1)\eta_{a_2}(t_2)\right]Cov\left[\delta_{a_1}(t_1), \delta_{a_2}(t_2)\right] & E\left[\eta_{a_1}(t_1)\eta_{a_2}(t_2)\right]Cov\left[\delta_{a_1}(t_1), \hbar_{a_2}(t_2)\right]\\ E\left[\eta_{a_1}(t_1)\eta_{a_2}(t_2)\right]Cov\left[\hbar_{a_1}(t_1), \delta_{a_2}(t_2)\right] & E\left[\eta_{a_1}(t_1)\eta_{a_2}(t_2)\right]Cov\left[\hbar_{a_1}(t_1), \hbar_{a_2}(t_2)\right]\end{Bmatrix}$$

$$= \begin{Bmatrix} p_{a_1a_2}Cov\left[\delta_{a_1}(t_1), \delta_{a_2}(t_2)\right] & p_{a_1a_2}Cov\left[\delta_{a_1}(t_1), \Gamma + J(\tau)\delta_{a_2}(t_2)\right]\\ p_{a_1a_2}Cov\left[\Gamma + J(\tau)\delta_{a_1}(t_1), \delta_{a_2}(t_2)\right] & p_{a_1a_2}Cov\left[\Gamma + J(\tau)\delta_{a_1}(t_1), \Gamma + J(\tau)\delta_{a_2}(t_2)\right]\end{Bmatrix}$$

$$= \begin{Bmatrix} p_{a_1a_2}\vartheta_{\delta_{a_1}\delta_{a_2}}(t_1 - t_2) & p_{a_1a_2}\vartheta_{\delta_{a_1}\delta_{a_2}}(t_1 - t_2)J(\tau)^T\\ p_{a_1a_2}J(\tau)\vartheta_{\delta_{a_1}\delta_{a_2}}(t_1 - t_2) & p_{a_1a_2}J(\tau)\vartheta_{\delta_{a_1}\delta_{a_2}}(t_1 - t_2)J(\tau)^T\end{Bmatrix}$$

$$= p_{a_1a_2}\begin{bmatrix}\vartheta_{a_1a_2}(t_1 - t_2) & \vartheta_{a_1a_2}(t_1 - t_2)J(\tau)^T\\ J(\tau)\vartheta_{a_1a_2}(t_1 - t_2) & J(\tau)\vartheta_{a_1a_2}(t_1 - t_2)J(\tau)^T\end{bmatrix}$$

Resulting from that stability and independence,

$$Cov\left\{z_{a_1}(t_1), z_{a_2}(t_2)\right\} = p_{a_1a_2}\begin{bmatrix}\vartheta_{a_1a_2}(\lambda) & \vartheta_{a_1a_2}(\lambda)J(\tau)^T\\ J(\tau)\vartheta_{a_1a_2}(\lambda) & J(\tau)\vartheta_{a_1a_2}(\lambda)J(\tau)^T\end{bmatrix},$$

and

$$Cov\left\{z_{a_1}(t_1), z_{a_2}(t_2)\right\} = p_{a_1a_2}\begin{bmatrix}\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon & \int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon J(\tau)^T\\ J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon & J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\exp\{i\upsilon\lambda\}d\upsilon J(\tau)^T\end{bmatrix}.$$

Definition: Whishart Distribution: Let's pretend δ and $\hbar$ are vectors that statistically independent in R^k, and $Vec[\delta \;\; \hbar]$ is a normal vector in $2k$ dimension. When this occurs, we say that $\varpi = \delta + i\hbar$, the complex random vector, follows the complex normal distribution. Three parameters, Γ, ℓ, and V, characterize this distribution.

$$\Gamma = E(\varpi),\ \ell = E[(\varpi - \Gamma)(\varpi - \overline{\Gamma})^T],\ V = E[(\varpi - \Gamma)(\varpi - \Gamma)^T],$$

Where matrix transposition denoted by ϖ^T, complex conjugate denoted by $\overline{\varpi}$. In this case, Γ can be a vector that follows complex distribution, ℓ can be a definite positive Hermitian in a complex matrix of k dimension, and for V, the symmetric relations matrix. The matrix $\overline{\ell} - \overline{V}^T \ell^{-1} V$ is also non-negative finite since ℓ and V satisfy this condition. Covariance matrices of δ and $\hbar$ may be calculated from ℓ and V using the corresponding formulas.

$$G_{\delta\delta} \equiv E[(\delta - \Gamma_\delta)(\delta - \Gamma_\delta)^T] = \frac{1}{2}\mathrm{Re}[\ell + V],$$

$$G_{\delta\hbar} \equiv E[(\delta - \Gamma_\delta)(\hbar - \Gamma_\hbar)^T] = \frac{1}{2}\mathrm{Im}[-\ell + V],$$

$$G_{\hbar\delta} \equiv E[(\hbar - \Gamma_\hbar)(\delta - \Gamma_\delta)^T] = \frac{1}{2}\mathrm{Im}[\ell + V],$$

$$G_{\hbar\hbar} \equiv E[(\hbar - \Gamma_\hbar)(\hbar - \Gamma_\hbar)^T] = \frac{1}{2}\mathrm{Re}[\ell - V],$$

Furthermore, inversely

$$\ell = G_{\delta\delta} + G_{\hbar\hbar} + i(G_{\hbar\delta} - G_{\delta\hbar}), \ell = G_{\delta\delta} - G_{\hbar\hbar} + i(G_{\hbar\delta} - G_{\delta\hbar}).$$

Theorem3.2

Consider $z_a(t)$ denote missing values about a stable probabilistic process $[\delta_a(t) \;\; \hbar_a(t)]^T, a = 1,...,\min(n,m)$, let $\eta_a(t)$ denotes a randomly Bernoulli sequence satisfying (3.3) and (3.4). If we define $\partial_a^{(T)}(g)$ to be (3.1) and assume that $\Re_a^{(T)}(g)$ holds true, then the distribution of $\partial_a^{(T)}(g)$ is approximated by

$\partial_a^{(T)}(g) \cong$

$$N^{c}_{n+m}\left(\underline{0}, p_{a_1a_2}\begin{bmatrix}\int_R f_{a_1a_2}(\upsilon)\Psi^{(T)}_{a_1a_2}(g_1-\upsilon,g_2-\upsilon)d\upsilon & \int_R f_{a_1a_2}(\upsilon)J(\tau)^T\psi^{(T)}_{a_1a_2}(g_1-\upsilon,g_2-\upsilon)d\upsilon \\ \int_R J(\tau)f_{a_1a_2}(\upsilon)\psi^{(T)}_{a_1a_2}(g_1-\upsilon,g_2-\upsilon)d\upsilon & \int_R J(\tau)f_{a_1a_2}(\upsilon)\psi^{(T)}_{a_1a_2}(g_1-\upsilon,g_2-\upsilon)d\upsilon\end{bmatrix}\right), \quad (3.8)$$

Such that,

$$\psi^{(T)}_{a_1a_2}(g_1-\upsilon,g_2-\upsilon)=(2\pi)^{-1}\left[A^{(T)}_{a_1a_2}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re^{(T)}_{a_1}(t_1)\sum_{t_2=0}^{T-1}\Re^{(T)}_{a_2}(t_2)\times$$

$$\times\exp\{-i[(g_1-\upsilon)t_1-i(g_2-\upsilon)t_2]\} \qquad (3.9)$$

Proof

Given (3.1) and (3.5), It may be shown that

$$E\{\partial_a(t)\}=0\ , \qquad (3.10)$$

$$Cov\left\{\partial^{(T)}_{a_1}(g_1),\partial^{(T)}_{a_2}(g_2)\right\}=$$

$$=Cov\left\{\left[2\pi\sum_{t_1=0}^{T-1}\left(\Re^{(T)}_{a_1}(t_1)\right)^2\right]^{-1/2}\sum_{t_1=0}^{T-1}\Re^{(T)}_{a_1}(t_1)\ z_{a_1}(t_1)\exp\{-ig_1t_1\},\left[2\pi\sum_{t_2=0}^{T-1}\left(\Re^{(T)}_{a_2}(t_2)\right)^2\right]^{-1/2}\sum_{t_2=0}^{T-1}\Re^{(T)}_{a_2}(t_2)z_{a_2}(t_2)\exp\{-ig_2t_2\}\right\}$$

$$=(2\pi)^{-1}\left[A^{(T)}_{a_1a_2}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re^{(T)}_{a_1}(t_1)\exp\{-ig_1t_1\}\sum_{t_2=0}^{T-1}\Re^{(T)}_{a_2}(t_2)\exp\{ig_2t_2\}Cov\{z_{a_1}(t_1),z_{a_2}(t_2)\}$$

$$=p_{a_1a_2}(2\pi)^{-1}\left[A^{(T)}_{a_1a_2}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re^{(T)}_{a_1}(t_1)\exp\{-ig_1t_1\}\sum_{t_2=0}^{T-1}\Re^{(T)}_{a_2}(t_2)\exp\{ig_2t_2\}\times$$

$$\times\begin{bmatrix}\vartheta_{xx}(t_1-t_2) & \vartheta_{xy}(t_1-t_2)\\ \vartheta_{yx}(t_1-t_2) & J(\tau)\vartheta_{xx}(t_1-t_2)J(\tau)^T\end{bmatrix},$$

and

$$Cov\left\{\partial^{(T)}_{a_1}(g_1),\partial^{(T)}_{a_2}(g_2)\right\}=(2\pi)^{-1}\left[A^{(T)}_{a_1a_2}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re^{(T)}_{a_1}(t_1)\exp\{-ig_1t_1\}\sum_{t_2=0}^{T-1}\Re^{(T)}_{a_2}(t_2)\exp\{ig_2t_2\}\times$$

$$\times p_{a_1a_2}\begin{bmatrix}\int_{-\infty}^{\infty}f_{a_1a_2}(\upsilon)\exp\{i\upsilon(t_1-t_2)\}d\upsilon & \int_{-\infty}^{\infty}f_{a_1a_2}(\upsilon)\exp\{i\upsilon(t_1-t_2)\}d\upsilon J(\tau)^T\\ J(\tau)\int_{-\infty}^{\infty}f_{a_1a_2}(\upsilon)\exp\{i\upsilon(t_1-t_2)\}d\upsilon & J(\tau)\int_{-\infty}^{\infty}f_{a_1a_2}(\upsilon)\exp\{i\upsilon(t_1-t_2)\}d\upsilon J(\tau)^T\end{bmatrix}$$

$$= (2\pi)^{-1}\left[A_{a_1a_2}^{(T)}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re_{a_1}^{(T)}(t_1)\sum_{t_2=0}^{T-1}\Re_{a_2}^{(T)}(t_2)\times$$

$$\times p_{a_1a_2}\exp\{-ig_1t_1+ig_2t_2+i\upsilon t_1-i\upsilon t_2\}\begin{bmatrix}\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)d\upsilon & \int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)J(\tau)^T d\upsilon \\ J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)d\upsilon & J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)J(\tau)^T d\upsilon\end{bmatrix}$$

$$= (2\pi)^{-1}\left[A_{a_1a_2}^{(T)}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re_{a_1}^{(T)}(t_1)\sum_{t_2=0}^{T-1}\Re_{a_2}^{(T)}(t_2)\times$$

$$\times p_{a_1a_2}\exp\{-i[(g_1-\upsilon)t_1-i(g_2-\upsilon)t_2]\}\begin{bmatrix}\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)d\vartheta & \int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)J(\tau)^T d\upsilon \\ J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)d\vartheta & J(\tau)\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)J(\tau)^T d\upsilon\end{bmatrix}$$

$$= \begin{bmatrix}\Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4\end{bmatrix} \tag{3.11}$$

where

$$\Omega_1 = p_{a_1a_2}\int_{-\infty}^{\infty} f_{a_1a_2}(\upsilon)\left\{(2\pi)^{-1}\left[A_{a_1a_2}^{(T)}(0)\right]^{-1}\sum_{t_1=0}^{T-1}\Re_{a_1}^{(T)}(t_1)\sum_{t_2=0}^{T-1}\Re_{a_2}^{(T)}(t_2)\times\right.$$

$$\left.\times\exp\{-i[(g_1-\upsilon)t_1-i(g_2-\upsilon)t_2]\}\right\}d\upsilon$$

$$= p_{a_1a_2}\int_R f_{a_1a_2}(\upsilon)\psi_{a_1a_2}^{(T)}(g_1-\upsilon, g_2-\upsilon)d\upsilon\,,$$

In the same way

$$\Omega_2 = p_{a_1a_2}\int_R f_{a_1a_2}(\upsilon)J(\tau)^T\psi_{a_1a_2}^{(T)}(g_1-\upsilon, g_2-\upsilon)d\upsilon\,,$$

$$\Omega_3 = p_{a_1a_2}\int_R J(\tau)f_{a_1a_2}(\upsilon)\psi_{a_1a_2}^{(T)}(g_1-\upsilon, g_2-\upsilon)d\upsilon\,,$$

and

$$\Omega_4 = p_{a_1a_2}\int_R J(\tau)f_{a_1a_2}(\upsilon)J(\tau)^T\psi^{(T)}_{a_1a_2}(g_1-\upsilon, g_2-\upsilon)d\upsilon\ .$$

Equation (3.8) is derived from (3.10) and (3.11), thus completing the proof.

By fitting $g_1 = g_2 = g \quad , g_1, g_2, g \in R.$ into Eq. (3.11), we get the following corollary,

Corollary 3.1

The following property is hold for the dispersion of $\partial^{(T)}_a(g)$ if we define $\partial^{(T)}_a(g)$, $a = 1,2,...,\min(n,m)$, $g \in R$ to be as (3.1):

$$D\partial^{(T)}_a(g) = p_{aa}\begin{bmatrix} \int_R f_{aa}(g-h)\psi^{(T)}_{aa}(h)dh & \int_R f_{aa}(g-h)J(\tau)^T\psi^{(T)}_{aa}(h)dh \\ \int_R J(\tau)f_{aa}(g-h)\psi^{(T)}_{aa}(h)dh & \int_R J(\tau)f_{aa}(g-h)J(\tau)^T\psi^{(T)}_{aa}(h)dh \end{bmatrix}, \quad (3.12)$$

so

$$\psi^{(T)}_{aa}(g) = (2\pi)^{-1}\left[A^{(T)}_{aa}(0)\right]^{-1}\left|A^{(T)}_a(g)\right|^2 \quad ,$$

Where $A^{(T)}_a(g)$, $a = 1,2,...,\min(n,m)$, $g \in R$ is the assumption variable.

Proof

According to (3.11), we obtain

$$D\partial^{(T)}_a(g) = p_{aa}\begin{bmatrix} \int_R f_{aa}(\vartheta)\psi^{(T)}_{aa}(g-\vartheta)d\vartheta & \int_R f_{aa}(\vartheta)J(\tau)^T\psi^{(T)}_{aa}(g-\vartheta)d\vartheta \\ \int_R J(\tau)f_{aa}(\vartheta)\psi^{(T)}_{aa}(g-\vartheta)d\vartheta & \int_R J(\tau)f_{aa}(\vartheta)J(\tau)^T\psi^{(T)}_{aa}(g-\vartheta)d\vartheta \end{bmatrix},$$

This is when $g_1 = g_2 = g \quad , g \in R$ and $a_1 = a_2 = a$, $a = 1,...,\min(n,m)$.

Letting ., we get (3.12)thus , $g - \upsilon = h$

Theorem 3.3

Let $\psi^{(T)}_{aa}(g)$, $a = 1,...,\min(n,m)$ is the base for any $g \in R$ if and only if it meets the following conditions.

1. $\int_{-\infty}^{\infty}\psi^{(T)}_{aa}(g)dg = 1, a = 1,...,\min(n,m), \qquad g \in R$ (3.13)

2. $$\lim_{T\to\infty}\int_{-\infty}^{-r}\psi_{aa}^{(T)}(g)dg=\lim_{T\to\infty}\int_{r}^{\infty}\psi_{aa}^{(T)}(g)dg=0,\ \forall r>0, a=1,...,\min(n,m), g\in R. \quad (3.14)$$

3. $$\lim_{T\to\infty}\int_{-r}^{r}\psi_{aa}^{(T)}(g)dg=1,\quad \forall\ a=1,...,\min(n,m), r>0,\quad g\in R. \quad (3.15)$$

Theorem 3.4

It follows from theorem 3.3 that if the function of spectrum density $f_{aa}(\delta)$, $a=1,...,\min(n,m),\ \delta\in R$ is finite continually at point $\delta=g, g\in R$ and $\psi_{aa}^{(T)}(\delta)$, $a=1,...,\min(n,m),\ \delta\in R$ is a function meets its characteristics, then

$$\lim_{T\to\infty}D\partial_a^{(T)}(g)=p_{aa}\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix},\quad a=1,...,\min(n,m). \quad (3.16)$$

Proof

In order to demonstrate the validity of formula (3.16), we must prove the following.

$$\lim_{T\to\infty}\left| D\partial_a^{(T)}(g)-p_{aa}\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix}\right| =0,$$

We have from corollary 3.1

$$\left| D\partial_a^{(T)}(g)-p_{aa}\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix}\right| \le$$

$$\le p_{aa}\int_{-\infty}^{\infty}\left\| \begin{bmatrix} f_{aa}(g-h) & f_{aa}(g-h)J(\tau)^T \\ J(\tau)f_{aa}(g-h) & J(\tau)f_{aa}(g-h)J(\tau)^T\end{bmatrix}-\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix}\right\| \psi_{aa}^{(T)}(h)dh\le$$

$$\le p_{aa}\int_{-\infty}^{-r}\left| \begin{bmatrix} f_{aa}(g-h) & f_{aa}(g-h)J(\tau)^T \\ J(\tau)f_{aa}(g-h) & J(\tau)f_{aa}(g-h)J(\tau)^T\end{bmatrix}-\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix}\right| \psi_{aa}^{(T)}(h)dh+$$

$$+P_{aa}\int_{-r}^{r}\left| \begin{bmatrix} f_{aa}(g-h) & f_{aa}(g-h)J(\tau)^T \\ J(\tau)f_{aa}(g-h) & J(\tau)f_{aa}(g-h)J(\tau)^T\end{bmatrix}-\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix}\right| \psi_{aa}^{(T)}(h)dh+$$

$$+P_{aa}\int_{r}^{\infty}\left| \begin{bmatrix} f_{aa}(g-h) & f_{aa}(g-h)J(\tau)^T \\ J(\tau)f_{aa}(g-h) & J(\tau)f_{aa}(g-h)J(\tau)^T\end{bmatrix}-\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T\end{bmatrix}\right| \psi_{aa}^{(T)}(h)dh$$

$= B_1 + B_2 + B_3$.

Now that $f_{a_1a_2}(h)$ is a continuous function at point $h = g, a_1, a_2 = 1,...,\min(n,m), g \in R$, we can write

$$B_2 = P_{aa}\int_{-r}^{r}\left\| \begin{bmatrix} f_{aa}(g-h) & f_{aa}(g-h)J(\tau)^T \\ J(\tau)f_{aa}(g-h) & J(\tau)f_{aa}(g-h)J(\tau)^T \end{bmatrix} - \begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T \end{bmatrix} \right\| \psi_{aa}^{(T)}(h)dh$$

$$= p_{aa}\int_{-\delta}^{r}\left\| \begin{bmatrix} f_{aa}(g-h) - f_{aa}(g) & f_{aa}(g-h)J(\tau)^T - f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g-h) - J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g-h)J(\tau)^T - J(\tau)f_{aa}(g)J(\tau)^T \end{bmatrix} \right\| \psi_{aa}^{(T)}(h)dh \le$$

$$\le \zeta \int_{-r}^{r} \psi_{aa}^{(T)}(h)dh \le \zeta \int_{-\infty}^{\infty} \psi_{aa}^{(T)}(h)dh,$$

Thus $B_2 \le \zeta$. Therefore, B_2 is extremely small if any ζ is very small; hence, $B_2 = 0$. If $f_{aa}(g)\ a = 1,...,\min(n,m), g \in R$ is limited by a constant value M, then

$$B_1 \le 2L \int_{-\infty}^{-r} \psi_{aa}^{(T)}(h)dh \xrightarrow[T\to\infty]{} 0,$$

based on the property (3.14). similarly $B_3 \xrightarrow[T\to\infty]{} 0$, then,

$$\left\| D\partial_a^{(T)}(g) - p_{aa}\begin{bmatrix} f_{aa}(g) & f_{aa}(g)J(\tau)^T \\ J(\tau)f_{aa}(g) & J(\tau)f_{aa}(g)J(\tau)^T \end{bmatrix} \right\| \xrightarrow[T\to\infty]{} 0.$$

It completes up the proof for the theorem.

Lemma 3.1

If $\Re_a^{(T)}(t), t \in R$, $a = \overline{1,n}$ is a limited function of data window, has limited fluctuations, and has a value of zero outside of the interval $[0, T-1]$, then

$$\sum_{t=0}^{T-1} \Re_a^{(T)}(t) \sim \mathrm{T}\int_0^1 \Re_a^{(T)}(\lambda)\, d\lambda, \tag{3.17}$$

where,

$$\frac{1}{T}\sum_{t=0}^{T-1} \Re_a^{(T)}(t) \xrightarrow[T\to\infty]{} \int_0^1 \Re_a^{(T)}(\lambda)\, d\lambda,\ a = \overline{1,n},\ T = 1,2,... \tag{3.18}$$

Lemma 3.2

If $\Re_a^{(T)}(t), t \in R,\ a = \overline{1,r}$ satisfies the Lipschitz condition and is limited within a constant M,

$$\sum_{\lambda=0}^{T-1} \left|\Re_a^{(T)}(t+\lambda) - \Re_a^{(T)}(t)\right| \le \zeta|\lambda|, \tag{3.19}$$

then,

$$\left|\sum_{t=0}^{T-1} \Re_{a_1}^{(T)}(\lambda+t)\Re_{a_2}^{(T)}(t)\exp\{-igt\} - \sum_{t=0}^{T-1} \Re_{a_1}^{(T)}(t)\Re_{a_2}^{(T)}(t)\exp\{-igt\}\right| \le M\zeta|\lambda|, \tag{3.20}$$

For every constant value ζ, $\lambda = \overline{[-(T-1),(T-1)]}$ and $g \in [-\pi, \pi]$.

Lemma 3.3

If every $g_1, g_2 \in [-\pi, \pi],\ (g_1 - g_2) \ne (\mathrm{mod}\, 2\pi)$ and $\Re_a^{(T)}(t),\ t \in R$, $a = 1,\ldots,\min(n,m)$ are limited by a constant value M where the Lipschitz condition is satisfied (3.19), then:

$$Cov\left\{\partial_{a_1}^{(T)}(g_1), \partial_{a_2}^{(T)}(g_2)\right\} \le \frac{M\zeta}{2\pi\sqrt{\sum_{t_1,t_2=0}^{T-1}\left(\Re_{a_1}^{(T)}(g_1)\right)^2\left(\Re_{a_2}^{(T)}(g_2)\right)^2}} \times$$

$$\times\left\{\frac{1}{M\zeta|(g_1-g_2)/2|}\sum_{\varepsilon=-T+1)}^{T-1}\left|\vartheta_{a_1a_2}(\lambda)\right| + \sum_{\varepsilon=-T+1}^{T-1}\left|\vartheta_{a_1a_2}(\lambda)\right|[|\lambda|+1]\right\}, \tag{3.21}$$

For every $a_1, a_2 = 1,\ldots,\min(n,m)$.

Theorem 3.5

All $g_1, g_2 \in [-\pi, \pi],\ (g_1 - g_2) \ne (\mathrm{mod}\, 2\pi)$ and $\Re_a^{(T)}(t), t \in R$, $a = 1,\ldots,\min(n,m)$ is limited, and

$$\sum_{\tau=-\infty}^{\infty}[|\lambda|+1]\left|\vartheta_{a_1a_2}(\lambda)\right| < \infty, \tag{3.22}$$

So,

$$\lim_{T\to\infty} Cov\left\{\partial_{a_1}^{(T)}(g_1), \partial_{a_2}^{(T)}(g_2)\right\} = 0, \qquad (3.23)$$

for all $a_1, a_2 = 1, \ldots, \min(n, m)$.

Proof

Both Lemma 3.3 and Lemma 3.1 are used directly in the proof.

4. Applications

Our theoretical investigation will be applied to a climatological Example as detailed in the part that follows.

4.1 Investigating Temperature Level with Solar Irradiance

This research interprets a monthly series of data that illustrates the monthly average temperature records and solar irradiance in Libya. The records were obtained from the Libyan meteorological office from January 2012 until December 2021.

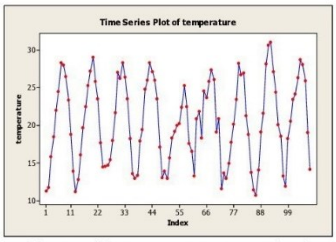
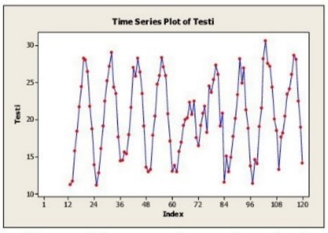
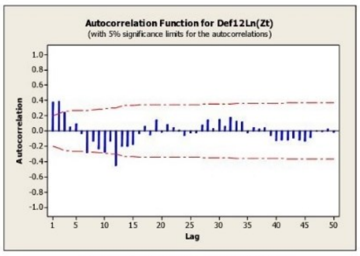
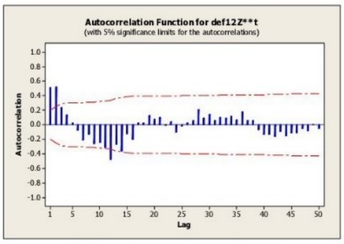
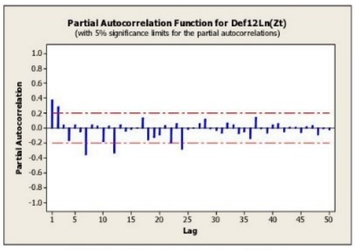
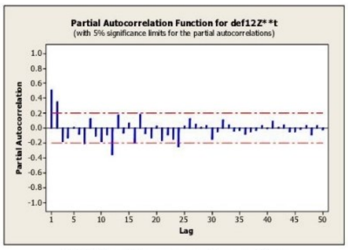
4.1.1 Temperature-Level Investigations

In this section, we shall compare the outcomes of our study of the survey model of a firmly stable time process (temperature Level) including a few missed data—against the outcomes from the classical setting, in which all data are available.

Let $H_a(t) = \eta_a(t)\delta_a(t)$ $, a = 1,2,\ldots,n$ be the quantity of interest, such $\delta_a(t), (t = 0, \pm 1, \ldots)$ is an firmly stability n – time vector process, and $\eta_a(t)$ is a Stochastic variables in Bernoulli sequence that are not dependent on $\delta_a(t)$ satisfying formulas (3.3) and (3.4). If we assume that $\delta_a(t), (t = (1,2,\ldots,T]$ is monthly mean of temperature values and that all of the values are obtainable, then $\eta = 1,\ H_a(t) = \delta_a(t),$ represents the classical case and assumes that some observations are missing at random (i.e., $\eta = 0,$).

Now, we comparison these research results with and without the missing data in table (4.1.1).

Table (4.1.1) Comparing Of Research Outcomes With and Without the Missing Values

Series without missed data	Series with missed data
 The monthly average temperatures level	 The monthly average temperatures level
 ACF of the seasonal variation	 ACF of the seasonal variation
 PACF of the seasonal variation	 PACF of the seasonal variation

Series without missed data

ARIMA Model: Temperature level

$ARIMA(2,0,0)\times(0,1,1)12$

Final Estimates

Type	Co-ef.	SE Co-ef.	T	P
AR 1	0.3447	0.1035	3.33	0.001
AR 2	0.1521	0.1030	1.49	0.142
SMA 12	0.8449	0.0838	10.08	0.000
Constant	0.002437	0.002640	0.93	0.356

Differencing: 0 regular, 1 seasonal of order 12

Number of values: Original series 108, after differencing 96

Residuals: SS = 1.03223 (back forecasts excluded)

MS = 0.01120 DF = 92

Modified Box-Pierce (Ljung-Box) Chi-Square statistic

Lag	12	24	36	48
Chi-Square	15.1	26.0	36.5	41.5
DF	8	20	32	44
P-Value	0.057	0.203	0.269	0.585

Series with missed data

ARIMA Model: Temperature level

$ARIMA(2,0,0)\times(0,1,2)12$

Final Estimates

Type	Co-ef.	SE Co-ef.	T	P
AR 1	0.5701	0.1059	5.38	0.000
AR 2	-0.0154	0.1058	-0.16	0.883
SMA 12	1.5640	0.1034	15.11	0.000
SMA 24	-0.6503	0.1015	-6.40	0.000
Constant	0.0011288	0.0009347	1.21	0.231

Differencing: 0 regular, 1 seasonal of order 12

Number of values: Original series 108, after differencing 96

Residuals: SS = 0.775841(back forecasts excluded)

MS = 0.008525 DF = 91

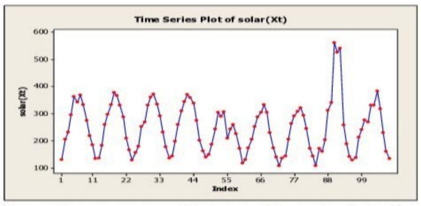
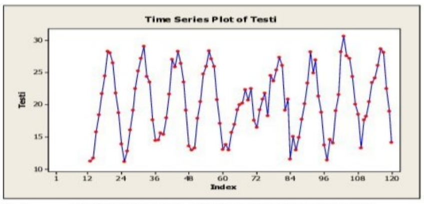
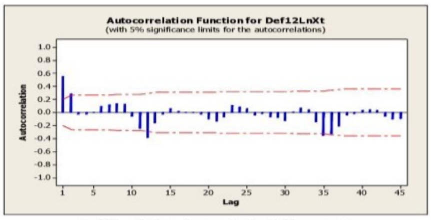
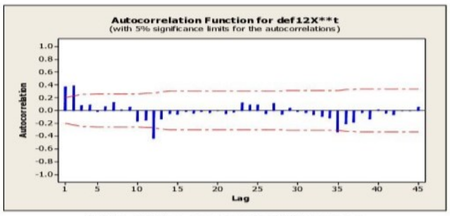
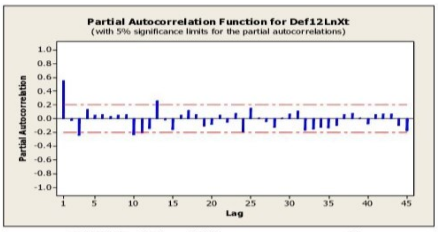
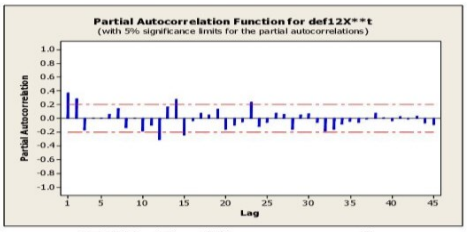
Modified Box-Pierce (Ljung-Box) Chi-Square statistic

Lag	12	24	36	48
Chi-Square	10.6	20.9	44.2	53.5
DF	7	19	31	43

4.1.2 Studying The Solar Irradiance

Using a firmly stable model of time process (Solar irradiance), we will compare our results—where some data are missed—to the classical results, where all values are recorded. We assume that the data $\hbar_a(t), t=[1,2,...,T]$ is the value of the average monthly records solar irradiance, thus all records are unmissed, $\eta=1, \quad H_a(t)=\hbar_a(t),$ which would be the standard model, now we further assume that there are some missed data in a randomly, i.e. $\eta=0,$, so that $\eta_a(t)$ is a Bernoulli series of stochastic variable that does not depend on $\hbar_a(t)$ in any predictable way, satisfying equations (3.3) and (3.4). The results of this comparison, with and without missing values, are shown in table (4.1.2).

Table (4.1.2) Observing the Differences between the Outcomes with and Without the Missing Solar Irradiance.

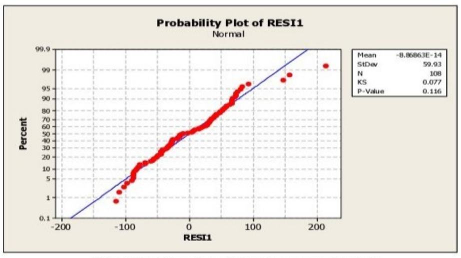
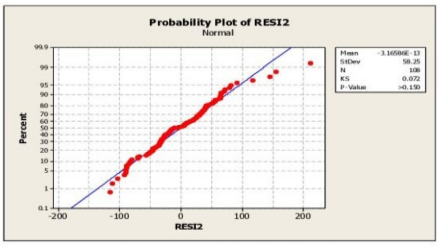
<table>
<tr><th>Series without missing values</th><th>Series with missing values</th></tr>
<tr><td>
1.1.2. The average of the monthly solar irradiance</td><td>
1.1.1. The average of the monthly solar irradiance</td></tr>
<tr><td>
ACF of the seasonal differences</td><td>
ACF of the seasonal differences</td></tr>
<tr><td>
PACF of the difference seasonally</td><td>
PACF of the difference seasonally</td></tr>
<tr><td>ARIMA Model: solar irradiance without missed data
$ARIMA(3,0,0)\times(0,1,2)12$
Final Estimates of Parameters
Type Co-ef SE Co- ef T P
AR 1 0.6444 0.1036 6.23 0.000
AR 2 0.1998 0.1219 1.65 0.106
AR 3 -0.3298 0.1005 -3.28 0.001
SMA 12 0.9058 0.1118 8.08 0.000
SMA 24 -0.0571 0.1638 -0.36 0.728
Constant -0.1523 0.8092 -0.18 0.852
Differencing: 0 regular, 1 seasonal order 12</td><td>ARIMA Model: solar irradiance with missed values
$ARIMA(3,0,0)\times(0,1,2)12$
Final Estimates for Parameters
Type Co- ef SE Co- ef T P
AR 1 0.5508 0.0975 5.66 0.000
AR 2 0.4378 0.1045 4.18 0.000
AR 3 -0.4198 0.0971 -4.34 0.000
SMA 12 1.0428 0.1088 9.59 0.000
SMA 24 -0.2016 0.1578 -1.29 0.206
Constant -0.0519 0.7479 -0.06 0.946
Differencing: 0 regular, 1 seasonal order 12</td></tr>
<tr><td>Number of records: Original series 108, after differencing 96
Residuals: SS = 113198 (back forecasts excluded)
MS = 1259 DF = 90
Modified Box-Pierce (Ljung-Box) Chi-Square statistic
Lag 12 24 36 48
Chi-Square 10.0 16.0 36.1 44.8
DF 6 18 30 42
P-Value 0.124 0.588 0.241 0.359</td><td>Number of data: Original series 108, after differencing 96
Residuals: SS = 122928 (back forecasts excluded)
MS = 1367 DF = 90
Modified Box-Pierce (Ljung-Box) Chi-Square statistic
Lag 12 24 36 48
Chi-Square 8.6 21.4 41.2 51.3
DF 6 18 30 42
P-Value 0.207 0.267 0.086 0.16</td></tr>
</table>

4.1.3 Examining the Linear Regression of Solar Irradiance and Average Temperature

This section adjusted the model of linear regression that representing the significant relationship between the rate monthly of solar radiation (watt / m^2) and the monthly average records of temperature level from year 2012 to year 2021. Our results, which represented for some missed data, will be compared to the classical results, which assume that all values are existent.

Let $\varpi(t)=[\delta(t)\quad \hbar(t)]^T$, where $\delta(t)$ is a mean records of temperature time process and $\hbar(t)$ is an average solar radiation time process; we initially assume that all records are obtainable ($\eta=1,$), and then we consider that some observations are missing ($\eta=0,$) at random. Table 4.1.3 displays our study results.

Table (4.1.3) the Results Are Compared Of Regression Analysis Outcomes With and Without Missing Data

Without missing data	With missing data
The equation of regression is solar Irradiance = - 10.4 + 12.8 Temperature level Predictor Co-ef SE Co-ef T P Constant -10.43 22.37 -0.48 0.643 temperature 12.886 1.054 12.03 0.000 S = 60.2158 R-Sq = 57.8 % R-Sq (adj) = 57.8% Analysis of Variance Source DF SS MS F P Regression 1 523988 523988 144.50 0.000 Residual Error 107 384353 3627 Total 108 908341	The equation of regression is Solar Irradiance = - 9.8 + 12.8 Temperature level Predictor Co-ef SE Co-ef T P Constant -9.84 22.71 -0.44 0.662 temperature 12.826 1.075 11.85 0.000 S = 58.5264 R-Sq = 57.1% R-Sq(adj) = 56.1% Analysis of Variance Source DF SS MS F P Regression 1 480535 480535 140.29 0.000 Residual Error 107 363086 3426 Total 108 843621
 Standardized residuals normal plot	 Normalized plot of residuals

4.1.4. Conclusion

1. The investigation the standard time process and time process with missed values produced the same outcomes. (see Tables 4.1.1 and 4.1.2).
2. As can be seen in Table 4.1.3, the results of the study of the model of linear regression between the averages monthly of solar irradiation and the monthly average temperature level with a few missed values were identical to those of the study of the traditional linear regression model.

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الخصائص التقريبية للسلاسل الزمنية المنفصلة ثنائية المتغير مع البيانات المفقودة

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الملخص:

يعد تحويل فوريه المحدود الممتد طريقة رياضية فعالة لتحليل بيانات السلاسل الزمنية بقيم المتجهات. في هذه الدراسة، يتم تطبيق التحويل على بيانات السلاسل الزمنية (n + m)، ويتم استخدام التقريبات التي تم الحصول عليها لإنشاء ميزات قابلة للاستخدام لمزيد من التحليل. قد تكون هذه التقنية مفيدة في مجال علم المناخ، حيث يمكن أن تشكل البيانات المفقودة تحديًا كبيرًا. قد يحلل الباحثون بيانات المناخ بشكل أكثر دقة باستخدام تحويل فوريه المحدود الممتد، حتى عندما تكون بعض الملاحظات مفقودة بشكل عشوائي. يمكن أن يؤدي ذلك إلى فهم أفضل لأنماط واتجاهات المناخ بمرور الوقت، وهو أمر ضروري للتنبؤ بالتغيرات المستقبلية وتطوير سياسات التخفيف الفعالة. بشكل عام، يعد تحويل فورييه المحدود الممتد تطورًا مثيرًا للاهتمام في مجال تحليل السلاسل الزمنية، مع العديد من التطبيقات الممكنة في مجموعة متنوعة من المجالات. يجب أن نتوقع تطورات أكثر إثارة في السنوات القادمة حيث يواصل الأكاديميون استكشاف قوتها وإتقان منهجياتها.

الكلمات المفتاحية: السلاسل الزمنية المستقرة غير المستمرة؛ نوافذ البيانات، تحويل فوريير، البيانات المفقودة، توزيع ويشارت.