The odd log- logistic Power Inverse Lindley distribution: Model, Properties and Applications

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The odd log- logistic Power Inverse Lindley distribution: Model, Properties and Applications

Dr. Mahmoud Eltehiwy and Dr. Mohamed Hamouda

Abstract

In this article, we introduce a new three-parameter odd log-logistic power inverse Lindley distribution and discuss some of its properties. These include the shapes of the density and hazard rate functions, mixture representation, the moments, the quantile function, and order statistics. Maximum likelihood estimation of the parameters and their estimated asymptotic standard errors are derived. Three algorithms are proposed for generating random data from the proposed distribution. A simulation study is carried out to examine the bias and root mean square error of the maximum likelihood estimators of the parameters. An application of the model to three real data sets is presented finally and compared with the fit attained by some other well-known two and three-parameter distributions for illustrative purposes. It is observed that the proposed model has some advantages in analyzing lifetime data as compared to other popular models in the sense that it exhibits varying shapes and shows more flexibility than many currently available distributions.

Keywords: Lambert function, maximum likelihood estimation, order statistics, power inverse Lindley distribution, Stochastic ordering.

1. Introduction

"Survival and reliability analysis is a very important branch of statistics. It has many applications in many applied sciences, such as engineering, public health, actuarial science, biomedical studies, demography, and industrial reliability. The failure behavior of any system can be considered as a random variable due to the variations from one system to another resulting from the nature of the system. Therefore, it seems logical to find a statistical model for the failure of the system. In other applications, survival data are categorized by their hazard rate, e.g., the number of deaths per unit in a period of time. The modeling of survival data depends on the behavior of the hazard rate. The hazard rate may belong to the monotone (non-increasing and non-decreasing hazard rate) or non-monotone (bathtub and upside-down bathtub [UBT] or unimodal hazard rate). Several lifetime models have been suggested in statistics literature to model survival data. The Weibull distribution is one of the most popular and widely used models in life testing and reliability theory. Lindley (1958) suggested a one-parameter distribution as an alternative model for survival data. This model is known as Lindley distribution. However, the Weibull and Lindley distributions are restricted when data shows non-monotone hazard rate shapes, such as the unimodal hazard rate function (Almalki and Nadarajah 2014; Almalki and Yuan 2013)."

"There are several real applications where the data show the non-monotone shape for their hazard rate. For example, Langlands et al. (1997) studied the data of 3878 cases of breast carcinoma seen in Edinburgh from 1954 to 1964 and noticed that mortality was initially low in the first year, reaching a peak in the subsequent years, and then declining slowly. Another real problem was analyzed by Efron (1988) who, using head and neck cancer data, found the hazard rate initially increased, reached a maximum, and decreased before it finally stabilized due to therapy. The inverse versions of some existing probability distributions, such as inverse Weibull, inverse Gaussian, inverse gamma, and inverse Lindley, show non-monotone shapes for their hazard rates; hence, we were able to model a non-monotone shape data. Erto and Rapone (1984) showed that the inverse Weibull distribution is a good fit for survival data, such as the time to breakdown of an insulating fluid subjected to the action of constant tension. The use of Inverse Weibull was comprehensively described by Murthy et al. (2004). Glen (2011) proposed the inverse gamma distribution as a lifetime model in the context of reliability and survival studies. Recently, a new upside-down bathtub-shaped hazard rate model for survival data analysis was proposed by Sharma et al. (2014) by using transmuted Rayleigh distribution. Sharma et al. (2015) introduced the inverse Lindley distribution as a one-parameter model for a stress-strength reliability model. Sharma et al. (2016) generalized the inverse Lindley into a two-parameter model called “the generalized inverse Lindley distribution.” Finally, a new reliability model of inverse gamma distribution referred to as “the generalized inverse gamma distribution” was proposed by Mead (2015), which includes the inverse exponential, inverse Rayleigh, inverse Weibull, inverse gamma, inverse Chi square, and other inverse distributions."

"Lindley (1958) proposed the Lindley distribution in the context of the Bayes theorem as a counter example of fiducial statistics with the probability density function (pdf)

$$f(y;\beta)=\frac{\beta^2}{1+\beta}(1+y)e^{-\beta y};\ \ y,\beta>0. \tag{1}$$

Ghitany et al. (2008) discussed the Lindley distribution and its applications extensively and showed that the Lindley distribution is a better fit than the exponential distribution based on the waiting time at the bank for service. The Lindley distribution has been extended by different researchers including Zakerzadeh and Dolati (2009), Nadarajah et al. (2011), Shanker and Mishra (2013), Ghitany et al. (2013), Ashour and Eltehiwy (2015), Eltehiwy (2019), Alizadeh et al. (2017). The inverse Lindley distribution was proposed by Sharma et al. (2015) using the transformation $X=\frac{1}{Y}$ with the pdf

$$f(x;\beta)=\frac{\beta^2}{1+\beta}\left(\frac{1+x}{x^3}\right)e^{-\frac{\beta}{x}}; \qquad \beta,x>0, \qquad (2)$$

where Y is a random variable having pdf (1)."

"Another two parameters inverse Lindley distribution introduced by Sharma et al. (2016), called "the generalized inverse Lindley distribution," is a new statistical inverse model for upside-down bathtub survival data that uses the transformation $X=Y^{-\frac{1}{\alpha}}$ with the pdf

$$f(x;\beta,\alpha)=\frac{\alpha\beta^2}{1+\beta}\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta}{x^\alpha}}; \qquad \beta,\alpha,x>0, \qquad (3)$$

ith Y being a random variable having pdf (1). Note that Barco et al. (2017) also obtained the generalized inverse Lindley distribution by taking the transformation $X=Y^{-\frac{1}{\alpha}}$ where Y follows inverse Lindley distribution known as power inverse Lindley distribution (PIL) with the same pdf."

"The pdf (3) can be shown as a mixture of two distributions as follows:

$$f(x;\beta,\alpha)=pf_1(x)+(1-p)f_2(x),$$

where,

$$p=\frac{\beta}{\beta+1}, \qquad f_1=\frac{\alpha\beta}{x^{\alpha+1}}e^{-\frac{\beta}{x^\alpha}},\ x>0 \ \text{ and } f_2=\frac{\alpha\beta^2}{x^{2\alpha+1}}e^{-\frac{\beta}{x^\alpha}},\ \ x>0.$$

We see that, PIL is a two-component mixture of inverse Weibull distribution (shape α and scale β) and generalized inverse gamma distribution (with shape parameters 2, α and scale β), with mixing proportion $p=\beta/(\beta+1)$."

Gleaton and Lynch (2004, 2006) introduced a new family of distributions which is called the Generalized log-logistic family of distributions. The cumulative distribution function (cdf) of this family is given by

$$FF(x;\theta,\xi)=\frac{G(x;\xi)^\theta}{G(x;\xi)^\theta+\overline{G}(x;\xi)^\theta}, \qquad (4)$$

where $\theta > 0$ is the shape parameter, $G(x;\xi)$ is the cdf of the baseline distribution, $\overline{G}(x;\xi) = 1 - G(x;\xi)$ is the survival function and ξ is the set of the parameters of the baseline distribution $G(.)$. In addition, the pdf of the family is

$$f(x;\theta,\xi) = \frac{\theta g(x;\xi) G(x;\xi)^{\theta-1} \overline{G}(x;\xi)^{\theta-1}}{\left[G(x;\xi)^{\theta} + \overline{G}(x;\xi)^{\theta}\right]^2}.$$

This family was called later the odd log-logistic family of distributions. If the baseline distribution possesses a closed form cdf, the generated new distribution will also possess a closed form cdf. One can easily show that

$$\frac{\log\left[\frac{F(x;\theta,\xi)}{\overline{F}(x;\theta,\xi)}\right]}{\log\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]} = \theta.$$

Therefore θ is the quotient of the log-odds ratio for the generated and baseline distributions."

"Now, by letting $G(x;\xi)$ in (4) to be the cdf of the power inverse Lindley distribution, where $\xi = (\beta,\alpha)$ is the set of parameters, we can obtain a new extension of the power inverse Lindley distribution, called the odd log-logistic power inverse Lindley (henceforth, OLL-PIL) distribution. The cdf, pdf and hazard rate function of this distribution are given by"

$$F(x;\alpha,\beta,\theta) = \frac{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta} e^{-\frac{\theta\beta}{x^{\alpha}}}}{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta} e^{-\frac{\theta\beta}{x^{\alpha}}} + \left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}}, \quad (5)$$

for $x > 0, \theta, \beta, \alpha > 0$ and the corresponding pdf is given by

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\theta\beta^2\left(\frac{1+x^{\alpha}}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^{\alpha}}}\left[\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)\right]^{\theta-1}\left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta-1}}{(1+\beta)\left\{\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)^{\theta} e^{-\frac{\theta\beta}{x^{\alpha}}} + \left[1-\left(1+\frac{\beta}{(1+\beta)x^{\alpha}}\right)e^{-\frac{\beta}{x^{\alpha}}}\right]^{\theta}\right\}^2}, \quad (6)$$

$$h(x;\alpha,\beta,\theta) = \frac{\alpha\theta\beta^2\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^\alpha}}\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)\right]^{\theta-1}}{(1+\beta)\left[1-\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]\left\{\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)^\theta e^{-\frac{\theta\beta}{x^\alpha}}+\left[1-\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^\theta\right\}}. \quad (7)$$

"We write $X\sim OLL-PIL(\alpha,\beta,\theta)$if the pdf of X can be written as (6). The new distribution is very flexible in the sense that it can be skewed and symmetric depending upon the specific choices of the parameters. Furthermore, the associated cdf is in closed form. Consequently, this distribution can be applied to modeling censored data too. This is a major motivation to carry out this work. Furthermore, in reliability engineering and lifetime analysis, we often assume that the failure times of the components within each system follow the exponential lifetimes; see, for example, Adamidis and Loukas (1998) among others and the references therein. This assumption may seem unreasonable because, for the exponential distribution, the hazard rate is a constant, whereas many real-life systems do not have constant hazard rates, and the components of a system are often more rigid than the system itself, such as bones in a human body, balls of a steel pipe, etc. Accordingly, it becomes reasonable to consider the components of a system to follow a distribution with a non-constant hazard function that has flexible hazard function shapes."

"An interpretation of the OLL-PIL distribution can be given as follows: Let X be a lifetime random variable having power inverse Lindley distribution. The odds ratio that an individual (or component) following the lifetime X will die (fail) at time x is $y = G(x;\alpha,\beta)/\overline{G}(x;\alpha,\beta)$. Here, one can consider this odds of death as a random variable, say Y. Now, if we model the randomness of the "odds of death" using the log-logistic distribution with scale parameter 1 and shape parameter θ, $\left(F_Y(y) = y^\theta/\left[1+y^\theta\right]\right)$ for $y > 0$. Then we can write

$$Pr(Y \le y) = F_Y\left(G(x;\alpha,\beta)/\overline{G}(x;\alpha,\beta)\right),$$

which is given by (5), see Cooray (2006) for more details regarding this interpretation.

Plots of the pdf are shown in Fig. 1. The pdfs appear always unimodal. The mode moves more to the right and the pdf becomes less peaked with increasing values of β. The mode moves more to the right and the pdf becomes less peaked with increasing values of θ. The pdf becomes more peaked with increasing values of. The behavior of $h(x)$ in (7) of the OLL-PIL for different values of the parameters α, β and θ are showed graphically in Fig. 2."

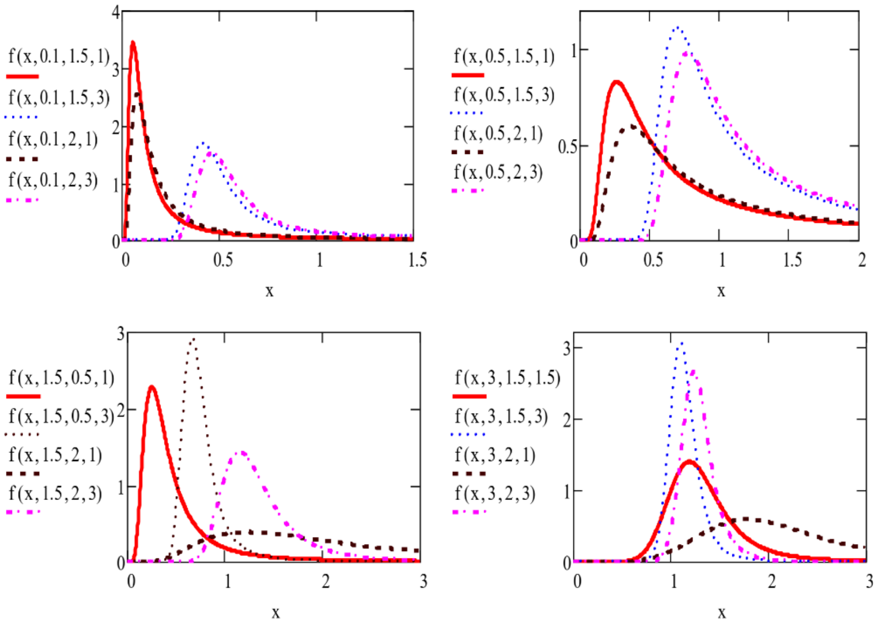

Figure 1. Pdfs of the OLL-PIL model for selected θ, β and α.

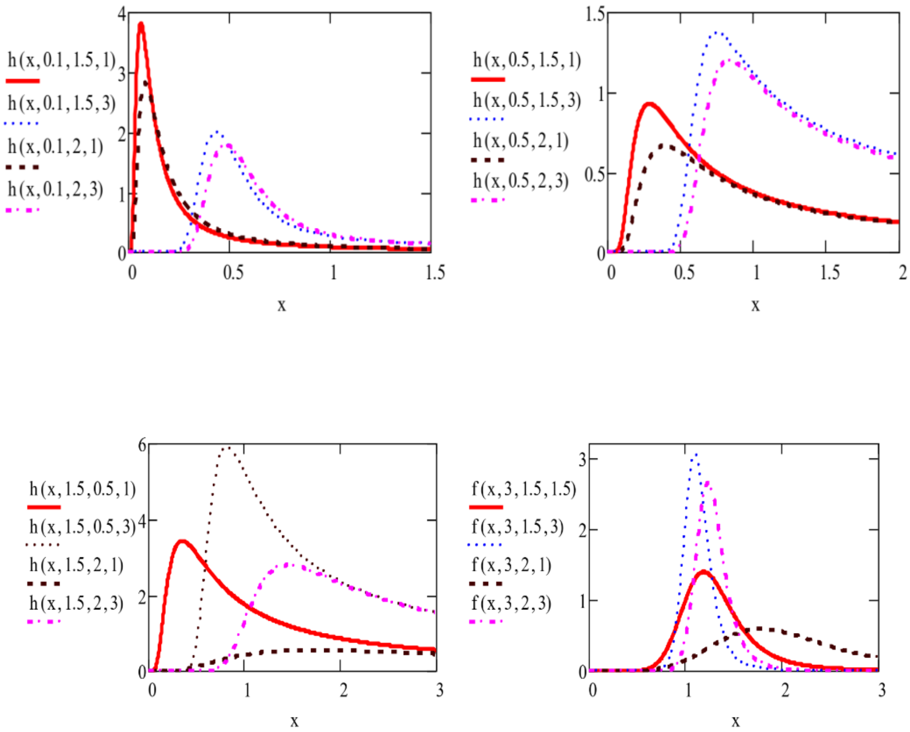

Figure 2. Hazard rate functions of the OLL-PIL model for selected θ, β and α.

"Because the hazard rate function of extended inverse Lindley distribution is always unimodel function in x, the new distribution is also a unimodal. Figure 2 illustrates the behavior of the hazard rate function of the OLL-PIL distribution at different values of the parameters involved. Concerning the hazard rate function of the odd log logistic power inverse Lindley distribution, which is shown in Fig. 2, it notably has the shape of an upside-down bathtub, therefore being unimodal in x.

This attractive flexibility makes the OLL-PIL hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations."

"We hope that this new distribution can be applied to describing lifetime data more properly than the existing distributions. The major motivation of introducing the OLL-PIL distribution can be summarized as follows. (i) The OLL-PIL distribution contains several lifetime distributions as special cases, such as the power inverse Lindley (PIL) distribution due to Barco et al. (2017) for $\theta = 1$: (ii) It is shown in Section 2 that the OLL-PIL distribution can be viewed as a mixture of exponentiated power inverse Lindley (EPIL) distributions introduced by Jan et al. (2018). (iii) The OLL-PIL distribution is a flexible model, which can be widely used for modeling lifetime data. (iv) The OLL-PIL distribution exhibits non-monotone hazard rates but does not exhibit a constant hazard rate, which makes this distribution to be superior to other lifetime distributions. (v) The OLL-PIL distribution outperforms several of the well-known lifetime distributions with respect to some real data examples."

Special cases:

- For $\theta = 1$, we obtain the power inverse Lindley distribution.
- For $\alpha = 1$, we obtain the odd log-logistic inverse Lindley distribution.
- For $\theta = \alpha = 1$, we obtain the inverse Lindley distribution.

"The rest of the article is organized as follows: In Section 2, we discuss some structural properties of the OLL-PIL distribution. Section 3 deals with the classical method of estimation (using maximum likelihood) of the model parameters of the OLL-PIL distribution. In Section 4, three real data sets are considered as an example to illustrate the applicability of OLL-PIL distribution. In Section 5, a simulation study is conducted to verify the efficacy of the said estimation procedure. In Section 6, we provide some concluding remarks."

2. Structural properties

In this section, we discuss some structural properties of the OLL-PIL distribution.

2.1 Mixture representations for the pdf and cdf

The EPIL distribution, introduced by Jan et al. (2018) has the pdf

$$f_{EPIL}(x;\alpha,\beta,\theta)=\frac{\alpha\theta\beta^2}{\beta+1}\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)e^{-\frac{\beta\theta}{x^\alpha}}\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)\right]^{\theta-1},\ x,\theta,\beta,\alpha>0 \quad (8)$$

We write $EPIL(\alpha,\beta,\theta)$ if the pdf of X can be expressed as (8). In addition, the cdf of the EPIL model is

$$F_{EPIL}(x;\alpha,\beta,\theta)=\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^\theta,\ x>0,\theta,\beta,\alpha>0 \quad (9)$$

Now, we show that the OLL-PIL distribution can be viewed as a mixture of EPIL distributions. Using the generalized binomial expansion, the numerator of (5) can be

$$\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)\right]^\theta e^{-\frac{\beta\theta}{x^\alpha}}=\sum_{k=0}^{\infty}a_k\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^k,$$

where $a_k=\sum_{j=k}^{\infty}-1^k\binom{\theta}{k}\binom{j}{k}$ and the denominator of (5) can be written as

$$\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)^\theta e^{-\frac{\theta\beta}{x^\alpha}}+\left[1-\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^\theta$$
$$=\sum_{k=0}^{\infty}b_k\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^k,$$

where $b_k=a_k+-1^k\binom{\theta}{k}$. Therefore, the cdf of the OLL-PIL distribution can be expressed as

$$F(x)=\frac{\sum_{k=0}^{\infty}a_k\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^k}{\sum_{k=0}^{\infty}b_k\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^k}=\sum_{k=0}^{\infty}c_k\left[\left(1+\frac{\beta}{(1+\beta)x^\alpha}\right)e^{-\frac{\beta}{x^\alpha}}\right]^k,$$

where $c_0 = \frac{a_0}{b_0} = 0$ and for $k \geq 1$ we have

$$c_k = b_0^{-1}[a_k - b_0^{-1}\sum_{r=1}^{k} b_r c_{k-r}].$$

Or equivalently, we can write the cdf of OLL-PIL as

$$F(x) = \sum_{k=1}^{\infty} c_k F_{EPIL}(x; k, \alpha, \beta) = \sum_{k=0}^{\infty} c_{k+1} F_{EPIL}(x; k+1, \alpha, \beta), \quad (10)$$

where $F_{EPIL}(x; k+1, \alpha, \beta)$ denotes the cdf of the EPIL distribution with parameters $k+1$, α and β. We note that $\sum_{k=0}^{\infty} c_{k+1} = 1$.

By differentiating equation (10), the pdf of the OLL-PIL distribution can be expanded as

$$f(x) = \sum_{k=0}^{\infty} c_{k+1} f_{EPIL}(x; k+1, \alpha, \beta), \quad (11)$$

where $f_{EPIL}(x; k+1, \alpha, \beta)$ denotes the pdf of the EPIL distribution with parameters $k+1$, α and β.

2.2 Moments

The r^{th} ordinary moment of X is given by

$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$. Then, using Eq.(11), we obtain

$$\mu_r' = (\beta)^{\frac{r}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{k}{i} c_{k+1} (k+1)^{\frac{r}{\alpha}} \frac{\left[i+1-\frac{r}{\alpha}+(k+1)\beta\right]\Gamma\left(i+1-\frac{r}{\alpha}\right)}{[(k+1)(\beta+1)]^{i+1}}$$

For r^{th} moment to exist, the constraint $\alpha > r$ must be satisfied.

The moment generating function $M_X(t) = E(e^{tx})$ of X can be derived from Eq. (11) as follows:

$$M_X(t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{n!} [\beta(k+1)]^{\frac{n}{\alpha}} \binom{k}{i} c_{k+1} \frac{\left[i+1-\frac{n}{\alpha}+(k+1)\beta\right]\Gamma\left(i+1-\frac{n}{\alpha}\right)}{[(k+1)(\beta+1)]^{i+1}}$$

2.3 Incomplete moments

"The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance, and medicine. The s^{th} incomplete moment, say say $\eta_s(t)$, of the OLL-PIL distribution is given by

$$\eta_s(t) = \int_0^t x^s f(x)dx,$$

$$\eta_s(t) = (\beta)^{\frac{s}{\alpha}} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{k}{i} c_{k+1} (k+1)^{\frac{s}{\alpha}} \frac{\left[i+1-\frac{s}{\alpha}+(k+1)\beta\right]\gamma\left(i+1-\frac{s}{\alpha},\frac{(k+1)(\beta+1)}{t^{\alpha}}\right)}{[(k+1)(\beta+1)]^{i+1}}, \quad (12)$$

where $\gamma(.,.)$ is the lower incomplete moments. “The first incomplete moment of the OLL-PIL distribution can be obtained by setting $s = 1$ in (12). The first incomplete moment is related to the Bonferroni and Lorenz curves, the mean residual, and mean waiting times. The Bonferroni and Lorenz curves are important in economics, reliability, demography, insurance, and medicine. The Lorenz curves, say $LO(x)$, and Bonferroni curve, say $BO(x)$, are defined by"

$$LO(x) = \frac{\eta_1(t)}{E(X)},$$

and

$$BO(x) = \frac{LO(x)}{F_{OLL-PIL}(x;\theta,\beta,\alpha)}.$$

2.4. Stochastic Orders

"Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. Suppose X_i is distributed according to (Eqs. 5 and 6) with common parameter β and parameters θ_i and α_i for $i = 1,2$. Let F_i denote the cumulative distribution of X_i and let f_i denote the probability density function of X_i."

A random variable X_1 is said to be smaller than a random variable X_2 in the

I. Stochastic order $(X_1 \leq_{st} X_2)$ if $F_1(x) \geq F_2(x)$ for all x.
II. Hazard rate order $(X_1 \leq_{hr} X_2)$if $h_1(x) \geq h_2(x)$ for all x.
III. Likelihood ratio order $(X_1 \leq_{Lr} X_2)$ if $\frac{f_1(x)}{f_2(x)}$ decreases in x.

"The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions"

$$X_1 \leq_{Lr} X_2 \Rightarrow X_1 \leq_{hr} X_2 \Rightarrow X_1 \leq_{st} X_2$$

The OLL-PILD is ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem:

Theorem 2.1. Let $X_1 \sim OLLPILD(\theta_1, \beta_1, \alpha_1)$ and $X_2 \sim OLL-PILD(\theta_2, \beta_2, \alpha_2)$. If $\beta_1 = \beta_2$, and $\theta_2 \geq \theta_1$ (or if $\beta_2 \geq \beta_1$ and $\theta_1 = \theta_2$), then $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$.

Proof. Straight forward and hence omitted.

Setting $\alpha_1 = \alpha_2$

Case 1: $\beta_1 = \beta_2$ and $\theta_2 \geq \theta_1$ we obtained $\frac{d}{dx}\left(\frac{f_2(x)}{f_1(x)}\right)$ as an increasing function of x.

Case 2: $\beta_1 \geq \beta_2$ and $\theta_2 = \theta_1$ we obtained $\frac{d}{dx}\left(\frac{f_2(x)}{f_1(x)}\right)$ as an increasing function of x.

This implies $X_1 \leq_{Lr} X_2$ and hence $X_1 \leq_{hr} X_2$ and $X_1 \leq_{st} X_2$."

2.5. Quantile Function

"Let X denotes a random variable with the probability density function (Eq. 6). The quantile function, say $Q(p)$, defined by $F\big(Q(p)\big) = p$ is the root of the equation

$$\left(1 + \frac{\beta}{(1+\beta)Q(p)^{\alpha}}\right) e^{-\frac{\beta}{Q(p)^{\alpha}}} = \frac{-(1+\beta)p^{1/\theta}}{p^{1/\theta} + (1-p)^{1/\theta}}, \tag{13}$$

for $0 < p < 1$. Multiplying (13) both sides by $e^{-1-\beta}$ we get,

$$-\left(1+\beta+\frac{\beta}{Q(p)^{\alpha}}\right)e^{-\left(1+\beta+\frac{\beta}{Q(p)^{\alpha}}\right)}=\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}$$

Using the Lambert W function which is the solution of the equation $W(z)e^{W(z)}$, where z is a complex number, we have"

$$W\left(\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}\right)=-\left(1+\beta+\frac{\beta}{Q(p)^{\alpha}}\right)$$

The negative Lambert W function of the real argument $\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}$ is

$$W_{-1}\left(\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}\right)=-\left(1+\beta+\frac{\beta}{Q(p)^{\alpha}}\right)$$

Which upon solving for $Q(p)$ results in

$$Q(p)=\left[-1-\frac{1}{\beta}-\frac{1}{\beta}W_{-1}\left(\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}\right)\right]^{-\frac{1}{\alpha}}.$$

Using above equation, the quartiles of the OLL-PIL distribution can be determined.

2.6. Asymptotic properties

Let X~OLL-PIL then the asymptotic of equation (5), 6) and (7) as $x \to 0$ are given by

$$F(x) \sim \left(\frac{\beta}{x^{\alpha}}\right)^{\theta} \quad \text{as } x \to 0$$

$$f(x) \sim \frac{\alpha\theta\beta^{\theta}}{x^{\alpha\theta+1}} \quad \text{as } x \to 0$$

$$h(x) \sim \frac{\alpha\theta\beta^{\theta}}{x^{\alpha\theta+1}} \text{ as } x \to 0$$

The asymptotic of equation (5), 6) and (7) as $x \to \infty$ are given by

$$1 - F(x) \sim \left(\frac{\beta}{1+\beta}\right)^{\theta} \frac{e^{-\frac{\theta\beta}{x^{\alpha}}}}{x^{\theta\alpha}} \quad \text{as } x \to \infty$$

$$f(x) \sim \theta\beta\alpha \left(\frac{\beta}{1+\beta}\right)^{\theta} \frac{e^{-\frac{\theta\beta}{x^{\alpha}}}}{x^{\alpha(\theta+1)+1}} \quad \text{as } x \to \infty$$

$$h(x) \sim \frac{\theta\beta\alpha}{x^{\alpha+1}} \text{ as } x \to \infty$$

"This attractive flexibility makes the OLL-PIL hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations."

2.7. Distribution of order statistics

"Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1, \dots, X_n$ are a random sample from an OLL-PIL distribution. Let $X_{i:n}$ denote the *i*-th order statistic. The pdf of $X_{i:n}$ can be expressed as (see Arnold et al., 1992)."

$$f_{i:n}(x) = K f(x) F^{i-1}(x)\{1 - F(x)\}^{n-i}$$

$$= K \sum_{j=0}^{n-i} (-1)^j \binom{n-j}{j} f(x) F(x)^{j+i-1}, \quad (14)$$

where $K = \frac{n!}{(i-1)!(n-i)!}$.

We use the result 0.314 of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n ($n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i} u^i,$$

where the coefficients $d_{n,i}$ (for $i = 1,2, \dots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (i a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i] a_m d_{n,i-m}.$$

We can demonstrate that the density function of the *i*-th order statistics of an OLL-PIL distribution can be expressed as

$$f_{i:n} = \sum_{r,k=0}^{\infty} \sum_{j=0}^{\infty} m_{r,k,j}^{*} f_{EPIL}(x, r+k+i+j, \alpha, \beta), \quad (14)$$

where $f_{EPIL}(x; \alpha, \beta, \theta)$ denotes the density of EPIL distribution with parameters α, β and θ and the coefficient $m^*_{r,k,j} \equiv m^*_{r,k,j}(i, n)'s$ are given by

$$m^*_{r,k,j} = \frac{n!(r+1)c_{r+1}(-1)^j a^*_{j+i-1.k}}{(i-1)!(n-i-j)!j!(r+k+i+j)},$$

In which the coefficients $c_r's$ are defined in subsection 2.1 and quantities $a^*_{j+i-1,k}$ can be determined such that $a^*_{j+i-1,0} = c_1^{j+i-1}$ and for $k \geq 1$

$$a^*_{j+i-1.k} = (kc_1)^{-1} \sum_{q=1}^{k} [q(j+i) - k] c_{q+1}\, a^*_{j+i-1,k-q}.$$

"Equation (14) is the main result of this section. It reveals that the pdf of the OLL-PIL order statistic is a linear combination of EPIL distributions. Therefore, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, and moment generating function, mean deviations and others can be derived using this result."

3. Maximum Likelihood Estimation of Parameters

Let $X_1, \ldots, X_n$ be a random sample of size n from OLL-PIL. Then, the log-likelihood function is given by

$$\mathcal{L}(\alpha, \beta, \gamma, \theta) = \sum_{i=1}^{n} \ln f(x_i),$$

$$= n[\ln(\alpha) + 2\ln(\beta) + \ln(\theta) - \ln(1+\beta)] + \sum_{i=1}^{n} \ln(1 + x_i^{\alpha})$$

$$-(2\alpha+1)\sum_{i=1}^{n} \ln(x_i) - \beta \sum_{i=1}^{n} x_i^{-\alpha} + (\theta - 1)\sum_{i=1}^{n} \ln[t_i(1-t_i)]$$

$$-2\sum_{i=1}^{n} \ln\left[t_i^{\theta} + (1-t_i)^{\theta}\right] \quad , \qquad (15)$$

where $t_i = \left(1 + \frac{\beta}{(1+\beta)x^{\alpha}}\right) e^{-\frac{\beta}{x^{\alpha}}}$.

The MLEs $\hat{\alpha}, \hat{\beta}, \hat{\theta}$ of α, β, θ are then the solutions of the following non-linear equations:

$$\frac{\partial}{\partial\alpha}\mathcal{L}(\alpha,\beta,\gamma,\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)}{x_i^{\alpha}+1} - 2\sum_{i=1}^{n}\ln(x_i) + \beta\sum_{i=1}^{n}x_i^{-\alpha}.\ln(x_i)$$

$$+(\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{t_i} + (1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\alpha)}}{1-t_i}$$

$$-2\theta\sum_{i=1}^{n}t_i^{(\alpha)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}} = 0, \qquad (16)$$

$$\frac{\partial}{\partial\beta}\mathcal{L}(\alpha,\beta,\gamma,\theta) = \frac{n(\beta+2)}{\beta(\beta+1)} - \sum_{i=1}^{n}x_i^{-\alpha} + (\theta-1)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{t_i} +$$

$$(1-\theta)\sum_{i=1}^{n}\frac{t_i^{(\beta)}}{1-t_i} - 2\theta\sum_{i=1}^{n}t_i^{(\beta)}\frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}} = 0 \qquad (17)$$

$$\frac{\partial}{\partial\theta}\mathcal{L}(\alpha,\beta,\gamma,\theta) =$$

$$\frac{n}{\theta} + \sum_{i=1}^{n}\ln[t_i(1-t_i)] - 2\sum_{i=1}^{n}\frac{t_i^{\theta}\ln(t_i)+(1-t_i)^{\theta}\ln(1-t_i)}{t_i^{\theta}+(1-t_i)^{\theta}} = 0 \qquad (18)$$

where

$$t_i^{(\alpha)} = \frac{\beta^2}{1+\beta}\left(\frac{1+x_i^{\alpha}}{x_i^{2\alpha+1}}\right)e^{-\frac{\beta}{x_i^{\alpha}}}\ \ln(x_i),$$

$$t_i^{(\beta)} = \frac{e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{\alpha}(1+\beta)^2} - \frac{e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{\alpha}}\left(\frac{\beta}{x_i^{\alpha}(1+\beta)}+1\right)$$

The above non-linear system of equations is solved by numerical iteration technique and maximum likelihood estimates are obtained.

For the three parameters OLL-PIL distribution, all the second order derivatives exist. Thus, we have the inverse dispersion matrix is

$$\begin{pmatrix}\hat{\theta}\\ \hat{\beta}\\ \hat{\alpha}\end{pmatrix} \sim N\left[\begin{pmatrix}\theta\\ \beta\\ \alpha\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33}\end{pmatrix}\right]$$

$$V^{-1} = -E\left[\begin{pmatrix}V_{11} & \cdots & V_{13}\\ \cdots & \cdots & \cdots\\ V_{31} & \cdots & V_{33}\end{pmatrix}\right] = -E\begin{pmatrix}\frac{\partial^2 \mathcal{L}}{\partial \theta^2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial \theta \partial \gamma}\\ \cdots & \cdots & \cdots\\ \frac{\partial^2 \mathcal{L}}{\partial \theta \partial \gamma} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial \gamma^2}\end{pmatrix}, \qquad (19)$$

Equation (19) is the variance covariance matrix of the $\mathrm{OLL}-\mathrm{PIL}\ (\theta, \beta, \alpha)$

$V_{11} = \frac{\partial^2 \mathcal{L}}{\partial \theta^2}$ $V_{12} = \frac{\partial^2 \mathcal{L}}{\partial \theta \partial \beta}$ $V_{13} = \frac{\partial^2 \mathcal{L}}{\partial \theta \partial \alpha}$ $V_{22} = \frac{\partial^2 \mathcal{L}}{\partial \beta^2}$,

$V_{23} = \frac{\partial^2 \mathcal{L}}{\partial \beta \partial \alpha}$ $V_{33} = \frac{\partial^2 \mathcal{L}}{\partial \alpha^2}$

The second derivatives of $\mathcal{L}$ is in Appendix.

By solving this inverse dispersion matrix, these solution will yield the asymptotic variance and co-variances of these ML estimators for $\hat{\theta}$, $\hat{\beta}$ and $\hat{\alpha}$. By using (Eq.19), approximately $100(1-\alpha)\%$ confidence intervals for θ, β, α and γ can be determined as

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}} \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}} \qquad \hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}}$$

where $Z_{\frac{\alpha}{2}}$ is the upper α-th percentile of the standand normal distribution."

4. Data Analysis

" In this section, we demonstrate the applicability of the OLL-PIL model for a real data. The data listed in Table 1 represents the average wind speed in Denmark reported in Hibatullah et al. (2018).

The applicability of EEILD is demonstrated by using some statistical tools suchs as Kolmogrov-Smirnov statistic, Akaike information criterion (AIC) defined by $-2\log L+2q$, Bayesian information criterion (BIC) defined by$-2\log L+q\log(n)$, where q is the number of estimated parameters and n is the sample size, and are compared with other distributions. AIC and BIC values estimates the quality of each model relative to each of the other models. The MLEs of the parameters are given in Table 3 and the statistical values mentioned above are computed and are given in Table 2. These values indicate that the proposed distribution fits well to the data compared to other tested distributions. The best model would be given by the highest value of $\log L$ and the lowest values of the AIC and BIC. Thus, the OLL-PIL distribution is compared with the Lindley (L) distribution, the power Lindley (PL) distribution, the inverse Lindley (IL) distribution, the power inverse Lindley (PIL) distribution, the Weibull (W) distribution, and the Gamma (G) distribution."

Table 1. The average wind speed in Denmark.

1.04525	2.28740	2.44529	2.68460	1.50003	3.33749
2.78426	4.79976	13.1893	5.45061	2.01266	1.27453
2.54918	1.32359	2.16495	1.32353	1.74341	2.29751
6.90446	1.71967	3.78884	1.48582	3.11761	3.26983
2.46577	3.52471	2.20266	5.10102	0.80668	2.65993
2.83905	0.38095	0.71543	3.00342	2.65187	4.53323
2.09819	10.9028	16.4941	1.77735	4.64156	5.73434
0.47927	1.38314	3.14792	4.88295	1.65586	2.09596
1.41378	1.89628	7.72747	0.80280	6.95507	1.52554
4.77888	1.03046	2.84926	5.02584	5.83996	2.71060

Table 2:Comparison criterion

Models	$\log L$	AIC	BIC	K-S statistic	P-value
Lindley	−129.586	261.172	263.266	0.0988	0.4087
Inverse Lindley	−132.532	267.063	269.157	0.1146	0.2423
Power Lindley	−129.022	262.044	266.233	0.0982	0.4156
Power Inverse Lindley	−129.671	263.343	267.531	0.1113	0.2723
Gamma (G)	−129.639	257.278	261.467	0.1012	0.3795
Weibull (W)	−128.960	259.920	266.109	0.0955	0.4507
OLL-PIL	**-123.611**	**253.222**	**259.506**	**0.0718**	**0.7582**

Table (3): Parameters MLES

Models	α	β	θ
Lindley (L)	----	0.49297	----
Inverse Lindley (IL)	1	2.50067	1
Power Lindley (PL)	1.09454	0.43377	1
Power Inverse Lindley (PIL)	1.26995	2.68507	1
Gamma (G)	1.95473	1.73284	---
Weibull (W)	1.33872	3.72481	----
OLL-PIL	0.22158	1.31936	7.162

The OLL-PIL takes the smallest K-S test statistic value and the largest value of its corresponding p-value. In addition, it takes the largest log likelihood. Therefore, OLL-PIL provides the best fit to this data.

5. Generation Algorithms and Monte Carlo Simulation Study

"In this section, the algorithms for generating random data from OLL-PIL distribution are given. A simulation study was also conducted to check the performance and accuracy of maximum likelihood estimates of the OLL-PIL model parameters." The Simulation study is performed using the statistical software Mathcad 14.

5.1 Generation algorithms

"In this subsection, different algorithms that can be used to generate random data from OLL-PIL distribution are presented.

Algorithm I. (mixture form of the inverse Lindley distribution)

1. Generate $U_i \sim uniform\ (0,1), i = 1, \dots, n$;

2. Generate $V_i \sim\ inverse\ Exponential\ (\beta), i = 1, \dots, n$;

3. Generate $G_i \sim inverse\ Gamma\ (2, \beta), i = 1, \dots, n$.

4. if $\frac{U_i^{1/\theta}}{U_i^{1/\theta}+(1-U_i)^{1/\theta}} \leq \frac{\beta}{1+\beta}$, then set $X_i = V_i^{1/\alpha}$, otherwise, set $X_i = G_i^{1/\alpha}, i = 1, \dots, n$.

Algorithm II. (mixture form of the Extended inverse Lindley distribution)

1. Generate $U_i \sim uniform\ (0,1), i = 1, \dots, n$;

2. Generate $Y_i \sim\ inverse\ Weibull\ (\alpha, \beta), i = 1, \dots, n$;

3. Generate $S_i \sim Generalized\ inverse\ Gamma\ (2, \alpha, \beta), i = 1, \dots, n$.

4. if $\frac{U_i^{1/\theta}}{U_i^{1/\theta}+(1-U_i)^{1/\theta}} \leq \frac{\beta}{1+\beta}$, then set $X_i = Y_i$, otherwise, set $X_i = S_i, i = 1, \dots, n$.

Algorithm III: (inverse CDF)

1. Generate $U_i \sim uniform(0,1), i = 1, \dots, n$;

2. Set

$$X_i = \left[-1 - \frac{1}{\beta} - \frac{1}{\beta} W_{-1}\left(\frac{-(1+\beta)p^{1/\theta}e^{-(1+\beta)}}{p^{1/\theta}+(1-p)^{1/\theta}}\right)\right]^{-\frac{1}{\alpha}}$$"

5.2 Monte Carlo simulation study

"In this subsection, we study the performance and accuracy of maximum likelihood estimates of the OLL-PIL model parameters by conducting various simulations for different combinations of 5 sample sizes with two sets of parameter values. Algorithm II was used to generate random data from the OLL-PIL distribution. The simulation study was repeated N = 10,000 times each with samples of size n = 25, 50, 100, 200, 400 combined with parameter values *(I)*: $\theta = 0.7, \beta = 4$, $\alpha = 0.8$, and *(II)*: $\theta = 1.5, \beta = 0.6, \alpha = 2$. Four quantities were computed in this simulation study: (i) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \alpha, \beta, \theta: \frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)$; (ii) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \alpha, \beta, \theta$: $\left[\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)^2\right]^{0.5}$;$(iii)$ Coverage probability (CP) of 95% confidence intervals of the parameter $\vartheta = \alpha, \beta, \theta$; (iv) Average width (AW) of 95% confidence intervals of the parameter $\vartheta = \alpha, \beta, \theta$. Table 4 presents the Average Bias, RMSE, CP and AW values of the parameters α, β and θ for different sample sizes. According to the results, it can be concluded that as the sample size n increases, the RMSEs decrease toward zero. We also observe that for all the parameters, the biases decrease as the sample size n increases. The results show that the coverage probabilities of the confidence intervals are quite close to the nominal level of 95% and that the average confidence widths decrease as the sample size increases. Consequently, the MLE's and their asymptotic results can be used for estimating and constructing confidence intervals even for reasonably small sample sizes.

Table 4: Monte Carlo simulation results: Average Bias, RMSE, CP and AW

		I				II			
Parameter	n	**Average bias**	**RMSE**	**CP**	**AW**	**Average bias**	**RMSE**	**CP**	**AW**
θ	25	0.649	0.388	0.961	0.784	0.656	0.866	0.944	3.841
	50	0.628	0.377	0.963	0.685	0.655	0.847	0.942	2.405
	100	0.592	0.361	0.964	0.472	0.652	0.841	0.945	1.876
	200	0.585	0.354	0.965	0.451	0.642	0.838	0.947	1.579
	400	0.575	0.335	0.974	0.365	0.571	0.797	0.963	0.367
β	25	2.449	2.198	0.963	1.238	0.587	0.693	0.963	5.050
	50	2.383	2.119	0.956	0.423	0.575	0.681	0.964	1.569
	100	1.926	2.176	0.957	0.329	0.556	0.621	0.968	0.847
	200	1.911	2.177	0.962	0.246	0.545	0.611	0.969	0.545
	400	1.848	1.986	0.964	0.203	0.442	0.495	0.970	0.254
α	25	0.663	0.744	0.943	2.179	0.626	0.953	0.922	1.882
	50	0.511	0.656	0.940	1.519	0.522	0.693	0.939	1.474
	100	0.448	0.499	0.931	1.147	0.435	0.595	0.926	0.839
	200	0.441	0.466	0.936	0.887	0.431	0.379	0.928	0.712
	400	0.427	0.441	0.946	0.339	0.349	0.343	0.949	0.419

6. Concluding Remarks

"In this paper, we have proposed a new family of distributions called odd log-logistic power inverse Lindley distribution. We get the probability density functions for odd log-logistic inverse Lindley and power inverse Lindley distributions as special cases from OLL-PIL. Some mathematical properties along with estimation issues are addressed. The hazard rate function behavior of the odd-logistic power inverse Lindley distribution shows that the subject distribution can be used to model reliability data. The estimation of parameters is approached by the method of maximum likelihood. We present a simulation study to exhibit the performance and accuracy of maximum likelihood estimates of the OLL-PIL model parameters. Real data application was also presented to illustrate the usefulness and applicability of the OLL-PIL distribution."

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Appendix

In this section, we report some needed derivatives in Section 3.

$$\frac{\partial^2 \mathcal{L}}{\partial \theta^2} = \frac{-n}{\theta^2} - 2\sum_{i=1}^{n} \frac{t_i^{\theta}(1-t_i)^{\theta}\ln(t_i)\ln\left(\frac{t_i}{1-t_i}\right) + t_i^{\theta}(1-t_i)^{\theta}\ln(1-t_i)\ln\left(\frac{1-t_i}{t_i}\right)}{\left[t_i^{\theta}+(1-t_i)^{\theta}\right]^2},$$

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\ln(x_i)^2}{\left(1+x_i^{\alpha}\right)^2} - \beta\sum_{i=1}^{n} \frac{(\ln x_i)^2}{x_i^{\alpha}} + (\theta-1)\sum_{i=1}^{n} \frac{t_i^{(\alpha\alpha)}t_i - \left[t_i^{(\alpha)}\right]^2}{t_i^2}$$

$$+(1-\theta)\sum_{i=1}^{n} \frac{t_i^{(\alpha\alpha)}(1-t_i) + \left[t_i^{(\alpha)}\right]^2}{(1-t_i)^2} - 2\theta\sum_{i=1}^{n} t_i^{(\alpha\alpha)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta(1-\theta)\sum_{i=1}^{n}\left[t_i^{(\alpha)}\right]^2 \frac{t_i^{\theta-2}+(1-t_i)^{\theta-2}}{t_i^{\theta}+(1-t_i)^{\theta}} +$$

$$2\theta^2\sum_{i=1}^{n}\left[t_i^{(\alpha)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}\right]^2.$$

$$\frac{\partial^2 \mathcal{L}}{\partial \beta^2} =$$

$$\left[\frac{-2n}{\beta^2} + \frac{n}{(\beta+1)^2}\right] + (\theta-1)\sum_{i=1}^{n} \frac{t_i^{(\beta\beta)}t_i - \left[t_i^{(\beta)}\right]^2}{t_i^2} +$$

$$(1-\theta)\sum_{i=1}^{n} \frac{t_i^{(\beta\beta)}(1-t_i) + \left[t_i^{(\beta)}\right]^2}{(1-t_i)^2} - 2\theta\sum_{i=1}^{n} t_i^{(\beta\beta)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}} -$$

$$2\theta(\theta-1)\sum_{i=1}^{n}\left[t_i^{(\beta)}\right]^2 \frac{t_i^{\theta-2}+(1-t_i)^{\theta-2}}{t_i^{\theta}+(1-t_i)^{\theta}} + 2\theta^2\sum_{i=1}^{n}\left[t_i^{(\beta)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}\right]^2.$$

.

$$\frac{\partial^2 \mathcal{L}}{\partial \theta \partial \alpha} = \sum_{i=1}^{n} \frac{t_i^{(\alpha)}}{t_i} - \sum_{i=1}^{n} \frac{t_i^{(\alpha)}}{1-t_i} - 2\sum_{i=1}^{n} t_i^{(\alpha)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta\sum_{i=1}^{n} \frac{t_i^{(\alpha)} t_i^{\theta-1}(1-t_i)^{\theta-1}\ln\left(\frac{t_i}{1-t_i}\right)}{\left[t_i^{\theta}+(1-t_i)^{\theta}\right]^2}.$$

$$\frac{\partial^2 \mathcal{L}}{\partial \theta \partial \beta} = \sum_{i=1}^{n} \frac{t_i^{(\beta)}}{t_i} - \sum_{i=1}^{n} \frac{t_i^{(\beta)}}{1-t_i} - 2\sum_{i=1}^{n} t_i^{(\beta)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$-2\theta \sum_{i=1}^{n} t_i^{(\beta)} \frac{t_i^{\theta-1}\ln(t_i)-(1-t_i)^{\theta-1}\ln(1-t_i)}{t_i^{\theta}+(1-t_i)^{\theta}}$$

$$+2\theta \sum_{i=1}^{n} t_i^{(\beta)} \frac{\left[t_i^{\theta}\ln(t_i)+(1-t_i)^{\theta}\ln(1-t_i)\right]\left[t_i^{\theta-1}-(1-t_i)^{\theta-1}\right]}{\left[t_i^{\theta}+(1-t_i)^{\theta}\right]^2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial\alpha\partial\beta} = \sum_{i=1}^{n} x_i^{-\alpha}\ln(x_i) + (\theta-1)\sum_{i=1}^{n} \frac{t_i^{(\alpha\beta)}t_i - t_i^{(\alpha)}t_i^{(\beta)}}{t_i^2}$$

$$+(1-\theta)\sum_{i=1}^{n} \frac{t_i^{(\alpha\beta)}(1-t_i)-t_i^{(\alpha)}t_i^{(\beta)}}{(1-t_i)^2} - 2\theta\sum_{i=1}^{n} t_i^{(\alpha\beta)} \frac{t_i^{\theta-1}-(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}} -$$

$$2\theta(\theta-1)\sum_{i=1}^{n} t_i^{(\alpha)}t_i^{(\beta)} \frac{t_i^{\theta-2}+(1-t_i)^{\theta-2}}{t_i^{\theta}+(1-t_i)^{\theta}} + 2\theta^2\sum_{i=1}^{n} t_i^{(\alpha)}t_i^{(\beta)} \left[\frac{t_i^{\theta-1}+(1-t_i)^{\theta-1}}{t_i^{\theta}+(1-t_i)^{\theta}}\right]^2$$

In which

$$t_i^{(\alpha\alpha)} = \frac{\beta^2}{1+\beta}\left(\frac{1}{x_i^{2\alpha+1}}\right) e^{-\frac{\beta}{x_i^{\alpha}}} [\ln(x_i)]^2[-(x^{\alpha}+2)+\beta(1+x^{-\alpha})],$$

$$t_i^{(\beta\beta)} = \frac{e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{2\alpha}}\left(\frac{\beta}{x_i^{\alpha}(1+\beta)}+1\right) - \frac{2e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{\alpha}(1+\beta)^3} - \frac{2e^{-\frac{\beta}{x_i^{\alpha}}}}{x_i^{2\alpha}(1+\beta)^2},$$

$$t_i^{(\alpha\beta)} = -\frac{\beta e^{-\frac{\beta}{x_i^{\alpha}}}\ln(x_i)(1+x_i^{\alpha})}{x_i^{2\alpha+1}(1+\beta)^2}\left[\frac{\beta(1+\beta)}{x^{\alpha}} - \beta - 2\right].$$

التوزيع الاحتمالى The odd log- logistic Power Inverse Lindley

النموذج وخصائصه وتطبيقاته

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الملخص:

في هذه المقالة، تم الحصول على توزيع جديد يسمى The **odd log- logistic Power Inverse Lindley**. تم اشتقاق العديد من الخصائص الإحصائية للتوزيع الجديد مثل العزوم، داله توليد العزوم، داله معدل الخطر، الترتيب العشوائي، الاحصاء الترتيبي. تم استخدام طريقة تقدير الإمكان الاعظم لتقدير المعلمات. وأخيرًا، تم تحليل قابلية تطبيق النموذج المقترح على البيانات الفعلية، كما تم إجراء مقارنة مع بعض التوزيعات الموجودة.

الكلمات المفتاحية:

دالة لامبرت، تقدير الإمكان الأعظم للمعلمات، الإحصاء الترتيبى، توزيع **Power Inverse Lindley**، الترتيب العشوائي.