The New Bivariate Odd Generalized Exponential Modified Weibull Distribution with Application

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The New Bivariate Odd Generalized Exponential Modified Weibull Distribution with Application

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Abstract

In this paper, we introduce a new bivariate odd generalized exponential modified Weibull (BOGEMW) distribution. This model includes some other well-known models such as the Weibull, generalized Weibull, exponential Weibull, Odd Generalized Exponential, and Modified Weibull distribution. The model introduced here is of Marshall and Olkin (1967) type. The marginals of the new bivariate distribution have odd generalized exponential modified Weibull distribution which proposed by Abdelall (2016). The joint probability density function and the joint cumulative distribution function are given in closed forms. Several properties of the new distribution are studied. The method of maximum likelihood is used for estimating the model parameters and the observed Fisher's information matrix is derived. We also conduct Monte Carlo simulation experiments to assess the finite sample properties of the proposed estimation methods. We prove empirically the importance and flexibility of the new model in modeling various types of data.

Keywords: Bivariate distribution, Maximum likelihood estimation, Modified Weibull distribution, Odd Generalized Exponential, Simulation.

1. Introduction

Modified Weibull distribution considered as another useful 3-parameter generalization of the Weibull distribution introduced by Lai et al. (2003). This distribution is capable of modeling a bathtub-shaped hazard-rate function and many real-life data that exhibit this property. Some well-known distributions such as the exponential, Rayleigh, linear failure rate and Weibull distributions are special cases of it. Tahir et al. (2015) proposed the odd generalized exponential (OGE) family. This method is flexible because the hazard rate shapes could be increasing, decreasing, bathtub and upside-down bathtub. Abdelall (2016) used this method to introduce five- parameters referred to as odd generalized exponential modified Weibull (OGEMW) distribution. The log-normal modified Weibull distribution and its reliability implications are introduced by Shakhatreh et al. (2019). However, there are other five-parameter generalizations to the modified Weibull some generalizations to the modified Weibull distribution and different methods of estimation are proposed by Shakhatreh et al. (2020).

In this paper we present a new distribution from the odd generalized exponential distribution and modified Weibull distribution called bivariate odd generalized exponential modified Weibull (BOGEMW) distribution, whose marginals are odd generalized exponential modified Weibull (OGEMW) distributions. It is obtained by using a method similar to that used to obtain Marshall-Olkin type. Several bivariate distributions are derived by using this method, Al-Khedhairi and El-Gohary (2008) presented a new class of bivariate Gompertz distribution of Marshall and Olkin type based on Gompertz and exponential distributions. Several properties of this are obtained and they found that the marginals distributions are not in known forms, Marshall and Olkin bivariate exponential distribution proposed by Sarhan and Balakrishnan (2007), the bivariate generalized exponential distribution proposed by Kundu and Gupta(2009), Marshall-Olkin bivariate Weibull distribution studied in Kundu and Gupta (2013), El-Sherpieny et al. (2013) proposed bivariate generalized Gompertz distribution whose marginals are generalized Gompertz distributions. A new bivariate exponentiated modified Weibull extension distribution introduced in El-Gohary et al. (2016). Mustafa and Mahmoud (2017) introduced a bivariate exponentiated modified Weibull distribution, whose marginals are exponentiated modified Weibull distribution and several properties of this distribution have been discussed. Babu and Jayakumar (2018) introduced bivariate modified Weibull distribution with modified Weibull distribution as marginals and discussed several properties. El-Morshedy et al. (2020) introduced a new five parameters bivariate discrete distribution, in the so-called the bivariate exponentiated discrete Weibull distribution and they found that the marginals are positive quadrant dependent. Eliwaa and El-Morshedy (2020) proposed a new generalized class for bivariate distributions, called bivariate odd Weibull-G family. Its marginal distributions are odd Weibull-G families. The family parameters have been estimated by using maximum likelihood and Bayesian methods. It is found that the Bayesian approach yields better estimates in terms of the MSE.

The rest of the paper is organized as follows. In Section 2, we describe the proposed model and discuss some different properties such as the joint cumulative distribution function, the joint probability density function, the marginal probability density functions and the conditional probability density functions of BOGEMW distribution. In Section 3 we introduced joint moment generating function of proposed bivariate distribution. Section 4 obtains some reliability measures. In Section 5 the maximum likelihood is used for estimating the parameters based on complete data. In Section 6 simulation study was carried out to assess the performance of the parameters. Finally, a numerical result is obtained using real data.

2. Bivariate Odd Generalized Exponential Modified Weibull Distribution

In this section, we introduce the BOGEMW distribution. We start with the joint cumulative function of the proposed bivariate distribution and it will be used it to derive the corresponding joint probability density function. Let Y be a random variable has univariate odd generalized exponential modified Weibull (OGEMW) distribution with parameters $\left(\underline{\Theta}_2 = (\beta, \theta, \gamma, \lambda)\right)$ and α, then the corresponding cumulative distribution function (CDF) is given by

$$F_Y\left(y; \alpha, \underline{\Theta_2}\right) = \left[1 - e^{-\lambda\left(e^{\theta y + \gamma y^{\beta}} - 1\right)}\right]^{\alpha}, y > 0, \alpha, \beta, \lambda, \theta, \gamma > 0, \quad (1)$$

and the probability density function (PDF) of OGEMW $\left(\underline{\Theta_2}\right)$ can be obtained as follows

$$f_Y\left(y; \alpha, \underline{\Theta_2}\right) = \quad \alpha\lambda(\theta + \beta\gamma y^{\beta-1}) e^{\theta y + \gamma y^{\beta}} e^{-\lambda\left(\theta y + \gamma y^{\beta} - 1\right)}$$

$$\times \left[1-e^{-\lambda(\theta y+\gamma y^{\beta}-1)}\right]^{\alpha-1} \tag{2}$$

Assume $U_i \sim$ OGEMW $(\alpha_i, \beta, \theta, \gamma, \lambda)$; $i = 1,2,3$ are three independent random variables. Define $Y_1 = \max\{U_1, U_3\}$ and $Y_2 = \max\{U_2, U_3\}$, then we say that the bivariate vector (Y_1, Y_2) has a bivariate odd generalized exponential modified Weibull, with parameters $(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma,\lambda)$. The corresponding joint cumulative distribution function (CDF) of the bivariate random vector can be expressed as follows

$$F_{Y_1,Y_2}(y_1,y_2) = \left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_1}\left[1-e^{-\lambda\left(e^{\theta y+\gamma y_2{}^{\beta}}-1\right)}\right]^{\alpha_2}$$

$$\times\left[1-e^{-\lambda\left(e^{\theta z+\gamma z^{\beta}}-1\right)}\right]^{\alpha_3}, \tag{3}$$

where $z = min\{y_1, y_2\}$. The corresponding joint probability density function (PDF) can be expressed as follows

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} f_1(y_1,y_2) & if \quad y_1 < y_2 \\ f_2(y_1,y_2) & if \quad y_2 < y_1 \\ f_0(y,y) & if \quad y_1 = y_2 = y \end{cases} \tag{4}$$

where

$$f_1(y_1,y_2) = f_{OGEMW}\left(y_1;\, \alpha_1+\alpha_3, \underline{\Theta_2}\right) f_{OGEMW}\left(y_2;\, \alpha_1+\alpha_3, \underline{\Theta_2}\right) \tag{5}$$

$$= \lambda(\alpha_1+\alpha_3)(\theta+\beta\gamma y_1{}^{\beta-1})e^{\theta y_1+\gamma y_1{}^{\beta}}e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}$$

$$\times\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3-1}\lambda\alpha_2(\theta+\beta\gamma y_2{}^{\beta-1})e^{\theta y_2+\gamma y_2{}^{\beta}}$$

$$\times e^{-\lambda\left(e^{\theta y_2+\gamma y_2{}^{\beta}}-1\right)}\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2{}^{\beta}}-1\right)}\right]^{\alpha_2-1}$$

$$f_2(y_1,y_2) = f_{OGEMW}\left(y_1;\, \alpha_1, \underline{\Theta_2}\right) f_{OGEMW}\left(y_2;\, \alpha_2+\alpha_3, \underline{\Theta_2}\right) \tag{6}$$

$$=\lambda\alpha_1\left(\theta+\beta\gamma {y_1}^{\beta-1}\right)e^{\theta y_1+\gamma {y_1}^{\beta}}e^{-\lambda\left(e^{\theta y_1+\gamma {y_1}^{\beta}}-1\right)}\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma {y_1}^{\beta}}-1\right)}\right]^{\alpha_1-1}\times$$

$$\lambda(\alpha_2+\alpha_3)\left(\theta+\beta\gamma {y_2}^{\beta-1}\right)e^{\theta y_2+\gamma {y_2}^{\beta}}e^{-\lambda\left(e^{\theta y_2+\gamma {y_2}^{\beta}}-1\right)}\times \left[1-e^{-\lambda\left(e^{\theta y_2+\gamma {y_2}^{\beta}}-1\right)}\right]^{\alpha_2+\alpha_3-1},$$

and

$$f_0(y,y)=\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}f_{OGEMW}\left(y;\alpha_1+\alpha_2+\alpha_3,\underline{\Theta_2}\right) \quad (7)$$

$$=\lambda\alpha_3\left(\theta+\beta\gamma y^{\beta-1}\right)e^{\theta y+\gamma y^{\beta}}e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\left[1-e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}.$$

Proof: The expressions $f_i(y_1,y_2)$, i=1,2 can be obtained by differentiating equation (3) with respect to y_i, i=1,2. But we can use the following fact to get $f_0(y)$

$$\int_0^{\infty}\int_0^{y_2}f_1(y_1,y_2)\,dy_1dy_2+\int_0^{\infty}\int_0^{y_1}f_2(y_1,y_2)\,dy_2dy_1+\int_0^{\infty}\int_0^{\infty}\int_0^{y_2}f_0(y)\,dy=1. \quad (8)$$

Let

$$I=\int_0^{\infty}\int_0^{y_2}f_1(y_1,y_2)\,dy_1dy_2 \;\; and \;\; II=\int_0^{\infty}\int_0^{y_1}f_2(y_1,y_2)\,dy_2dy_1.$$

Then,

$$\mathrm{I}=\int_0^{\infty}\left\{\lambda\alpha_2\left(\theta+\beta\gamma {y_2}^{\beta-1}\right)e^{\theta y_2+\gamma {y_2}^{\beta}}e^{-\lambda\left(e^{\theta y_1+\gamma {y_1}^{\beta}}-1\right)}\times\right.$$

$$\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_{1+}\alpha_2+\alpha_3-1}\Bigg\}dy_2, \qquad (9)$$

$$=\frac{\alpha_2}{\alpha_{1+}\alpha_2+\alpha_3},$$

similarly,

$$II=\int_0^{\infty}\int_0^{y_1}f_2(y_1,y_2)\,dy_2dy_1=\frac{\alpha_1}{\alpha_{1+}\alpha_2+\alpha_3} \qquad (10)$$

and

$$\int_0^{\infty}f_0(y,y)dy=1-\frac{\alpha_1}{\alpha_{1+}\alpha_2+\alpha_3}-\frac{\alpha_2}{\alpha_{1+}\alpha_2+\alpha_3}$$

$$=\frac{\alpha_3}{\alpha_{1+}\alpha_2+\alpha_3}.$$

Note that

$$\int_0^{\infty}f_0(y,y)dy=\alpha_3\int_0^{\infty}\left[\left[F_{OGEMW}\left(y;\ \underline{\Theta_2}\right)\right]^{\alpha_{1+}\alpha_2+\alpha_3-1}f_{OGEMW}\left(y;\ \underline{\Theta_2}\right)\right]dy$$

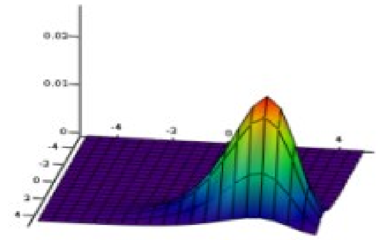
$$f_0(y,y)=\alpha_3\lambda(\theta+\beta\gamma y^{\beta-1})e^{\theta y+\gamma y^{\beta}}e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\times$$
$$\left[1-e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\right]^{\alpha_{1+}\alpha_2+\alpha_3-1}$$

$$=\frac{\alpha_3}{\alpha_{1+}\alpha_2+\alpha_3}\Bigg\{(\alpha_{1+}\alpha_2+\alpha_3)\lambda(\theta+\beta\gamma y^{\beta-1})e^{\theta y+\gamma y^{\beta}}e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\times$$
$$\left[1-e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\right]^{\alpha_{1+}\alpha_2+\alpha_3-1}\Bigg\}.$$

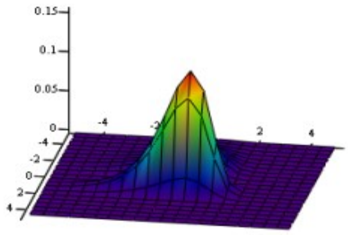
Thus

$$f_0(y,y)=\frac{\alpha_1}{\alpha_{1+}\alpha_2+\alpha_3}f_{OGEMW}\left(y;\alpha_1+\ \alpha_2+\ \alpha_3,\underline{\Theta_2}\right).$$

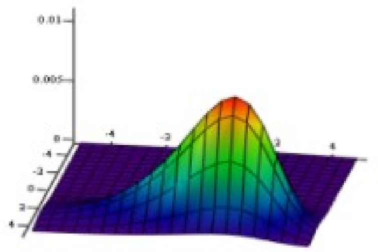
Figure (1) shows some plots of the PDF of BOGEMW model for some parameter values $\alpha_1, \alpha_2, \alpha_3, \beta, \theta, \gamma \; and \; \lambda$ to demonstrate its flexibility.



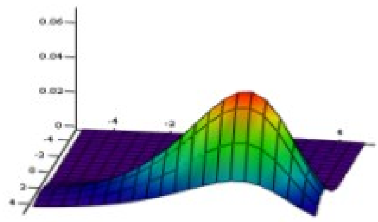
$(i)\ \beta = 1, \gamma = 0.1,\ \alpha_1 = 0.2,\ \alpha_2 = 0.2, \alpha_3 = 0.2, \lambda = 0.2, \theta = 0.3$



$(ii)\ \beta = 1, \gamma = 0.3,\ \alpha_1 = 0.1,\ \alpha_2 = 0.1,\ \alpha_3 = 0.1, \lambda = 0.3, \theta = 0.1$



$(iii)\ \beta = 0.2, \gamma = 0.2,\ \alpha_1 = 0.3,\ \alpha_2 = 0.3, \alpha_3 = 0.3, \lambda = 0.2, \theta = 0.5$



$(iv)\ \beta = 1, \gamma = 0.1,\ \alpha_1 = 0.1,\ \alpha_2 = 1, \alpha_3 = 1, \lambda = 0.3, \theta = 0.1$

Figure(1): Joint PDF of BOGEMW $(\alpha_1, \alpha_2, \alpha_3, \beta, \theta, \gamma, \lambda)$

The marginal probability density functions of Y_1 and $Y_2 > 0$ are univariate odd generalized exponential modified weibull distribution are given by

$$f_{Y_1}(y_1) = \lambda(\alpha_1 + \alpha_3)\left(\theta + \beta\gamma {y_1}^{\beta-1}\right) e^{\theta y_1 + \gamma {y_1}^{\beta}} e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)} \times \left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}\right]^{\alpha_1 + \alpha_3 - 1} \quad (11)$$

$$= f_{OGEMW}\left(y_1;\ \alpha_1 + \alpha_3, \theta, \lambda, \gamma, \beta\right),$$

and

$$f_{Y_2}(y_2) = \lambda(\alpha_2 + \alpha_3)\left(\theta + \beta\gamma {y_2}^{\beta-1}\right)e^{\theta y_2 + \gamma y_2^{\beta}} e^{-\lambda\left(e^{\theta y_2 + \gamma y_2^{\beta}} - 1\right)}$$

$$\times \left[1 - e^{-\lambda\left(e^{\theta y_2 + \gamma y_2^{\beta}} - 1\right)}\right]^{\alpha_2 + \alpha_3 - 1} \quad (12)$$

$$= f_{OGEMW}\left(y_2;\ \alpha_2 + \alpha_3, \theta, \lambda, \gamma, \beta\right).$$

Proof. First, we will derive $f(y_1)$ from the fact that

$$f_{Y_1}(y_1) = \int_0^{\infty} f\left(y_1, y_2\right) dy_2\,, for\ \ y_1, y_2 > 0$$

$$= \Phi\left(y_1\right) + \Psi\left(y_1\right) + \ f_0(y).$$

Where

$$\Phi\left(y_1\right) = \int_0^{y_1} f_2(y_1, y_2)\, dy_2 \ \text{ and }\ \Psi\left(y_1\right) = \int_{y_1}^{\infty} f_1(y_1, y_2)\, dy_2$$

By using the expressions of $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ in (5) and (6). We show that

$$\Phi\left(y_1\right) = \int_0^{\infty} f_2(y_1, y_2)\, dy_2$$

$$= \int_0^{y_1} f_{OGEMW}(y_1;\ \alpha_1, \theta, \lambda, \gamma, \beta)\, f_{OGEMW}(y_2;\ \alpha_2 + \alpha_3, \theta, \lambda, \gamma, \beta)\, dy_2$$

$$= f_{OGEMW}\left(y_1;\ \alpha_1, \theta, \lambda, \gamma, \beta\right) \int_0^{y_1} f_{OGEMW}(y_2;\ \alpha_2 + \alpha_3, \theta, \lambda, \gamma, \beta)\, dy_2\,.$$

Hence

$$\Phi\left(y_1\right) = \alpha_1\lambda\left(\theta + \beta\gamma {y_1}^{\beta-1}\right)e^{-\lambda\left(e^{\theta y_1 + \gamma y_1^{\beta}} - 1\right)} e^{\theta y_1 + \gamma y_1^{\beta}} \times$$

$$\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_1-1}\times\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_2+\alpha_3}$$

$$\int_0^{y_1}(\alpha_2+\alpha_3)\lambda\left(\theta+\beta\gamma y_2{}^{\beta-1}\right)e^{-\lambda\left(e^{\theta y_2+\gamma y_2{}^{\beta}}-1\right)}e^{\theta y_2+\gamma y_2{}^{\beta}}$$

$$\times\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2{}^{\beta}}-1\right)}\right]^{\alpha_1-1}$$

$$=\alpha_1\lambda\left(\theta+\beta\gamma y_1{}^{\beta-1}\right)e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}e^{\theta y_1+\gamma y_1{}^{\beta}}\times$$

$$\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}$$

Similarly

$$\Psi(y_1)=\int_{y_1}^{\infty}f_1(y_1,y_2)\,dy_2$$

$$=\int_0^{y_1}f_{OGEMW}(y_1;\,\alpha_1+\alpha_3,\theta,\lambda,\gamma,\beta)\,f_{OGEMW}(y_2;\,\alpha_2,\theta,\lambda,\gamma,\beta)\,dy_2\,.$$

It is equal to

$$\Phi(y_1)=\lambda(\alpha_1+\alpha_3)\left(\theta+\beta\gamma y_1{}^{\beta-1}\right)e^{\theta y_1+\gamma y_1{}^{\beta}}e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\times$$

$$\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3-1}\times\left\{1-\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_2}\right\}$$

and

$$f_0(y)=\lambda\alpha_3\left(\theta+\beta\gamma y^{\beta-1}\right)e^{\theta y+\gamma y^{\beta}}e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}$$

$$\times\left[1-e^{-\lambda\left(e^{\theta y+\gamma y^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_2+\alpha_3-1}$$

Then, the marginal pdf with $BOGEMW$

$$f_{Y_1}(y_1) = \Phi(y_1) + \Psi(y_1) + f_0(y)$$

After some calculations, we have been found that the marginal $f_{Y_1}(y_1)$

$$f_{Y_1}(y_1) = \lambda(\alpha_1 + \alpha_3)\left(\theta + \beta\gamma {y_1}^{\beta-1}\right)e^{\theta y_1 + \gamma {y_1}^{\beta}} e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}$$

$$\times \left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}\right]^{\alpha_1 + \alpha_3 - 1}$$

$$= f_{OGEMW}(y_1;\ \alpha_1 + \alpha_3, \theta, \lambda, \gamma, \beta)$$

Note that:

The marginal pdf of y_1 and y_2 can be derived in another way. For this, we first derive the marginal distribution function of y_1 as follows:

$$\text{F}(y) = p\left(Y_1 > y_1\right)$$

$$= p\left(min(U_1, U_3 > y_1)\right)$$

$$= p(U_1 > y_1)\, p(U_3 > y_1)$$

$$= \left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}\right]^{\alpha_1} \left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}\right]^{\alpha_3}$$

$$= \left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}\right]^{\alpha_1 + \alpha_3}$$

The marginal pdf of $y_1$is given as follows

$$f_{Y_1}(y_1) = \frac{\partial F(y_1)}{\partial y_1}$$

$$f_{Y_1}(y_1) = \lambda(\alpha_1 + \alpha_3)\left(\theta + \beta\gamma {y_1}^{\beta-1}\right)e^{\theta y_1 + \gamma {y_1}^{\beta}} e^{-\lambda\left(e^{\theta y_1 + \gamma {y_1}^{\beta}} - 1\right)}$$

$$\times\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3-1}$$

Similarly

$$f_{Y_2}(y_2)=\lambda(\alpha_2+\alpha_3)\left(\theta+\beta\gamma y_2^{\beta-1}\right)e^{\theta y_2+\gamma y_2^{\beta}}e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}$$
$$\times\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3-1}.$$

Thus, the conditional probability density functions of bivariate distributions are obtained from the joint and marginal distributions $f_{Y_1,Y_2}(y_1,y_2)$ *and* $f_{Y_2}(y_2)$ respectively.Thus, the conditional probability density functions of Y_1 for fixed values of Y_2 is

$$f_{Y_1|Y_2}(y_1|y_2)=\frac{f(y_1,y_2)}{f(y_2)},\qquad if\ \ f(y_2)>0$$

$$f_{Y_1|Y_2}(y_1|y_2)=\begin{cases} f_{y_1}(y_1|y_2) & if\quad y_1<y_2\\ f_{Y_1,Y_2}(y_1|y_2) & if\quad y_1>y_2\\ f_{Y_1,Y_2}(y_1|y_2) & if\quad y_1=y_2\end{cases}$$

where

$$f^{(1)}_{Y_1|Y_2}(y_1|y_2)=$$
$$\frac{\lambda(\alpha_2+\alpha_3)\left(\theta+\beta\gamma y_2^{\beta-1}\right)e^{\theta y_2+\gamma y_2^{\beta}}e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3-1}}{(\alpha_2+\alpha_3)\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_3}},$$

$$f^{(2)}_{Y_1|Y_2}(y_1|y_2)=\lambda\alpha_1\left(\theta+\beta\gamma y_1^{\beta-1}\right)e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}e^{\theta y_1+\gamma y_1^{\beta}}$$
$$\times\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_1-1},$$

and

$$f^{(3)}_{Y_1|Y_2}(y_1|y_2) = \frac{\alpha_3}{(\alpha_2+\alpha_3)}\left[1 - e^{-\lambda\left(e^{\theta y_1+\gamma y_1{}^{\beta}}-1\right)}\right]^{\alpha_1}$$

3. Moment generating function

In this section, we present the joint moment generating function of (Y_1, Y_2), the marginal moment generating function of Y_1 and we will use the moment marginal generating function of Y_1 to derive the expectation of Y_1.

3.1 The marginal moment generating function

We drive the marginal moment generating function of Y_1:

Lemma 1. If $y_1 \sim OGEMW\ \left(\alpha_1+\alpha_2,\ \underline{\Theta_2}\right)$,then the moment generating function of y_1 is given by

$M_{y_1}(t) = 2(\alpha_1 + \alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+j}(j+1)^k}{k!}\binom{\alpha_1+\alpha_3-1}{j}e^{(j+1)}\left[\frac{1}{(t-2-2k)}\right]$ (13)

Proof. Using $f_{Y1}(y_1)$ in (8) with $(\theta = \beta = \gamma = \lambda = 1)$ and substituting in the following equation

$$M_{y_1}(t) = E(e^{-ty_1}) = \int_0^{\infty} e^{-ty_1} f_{y_1}(y_1)\ dy_1,$$

We get

$$M_{y_1}(t) = 2(\alpha_1+\alpha_3)\int_0^{\infty} e^{-ty_1}e^{2y}e^{(e^{2y_1}-1)}\left[1-e^{(e^{2y_1}-1)}\right]^{\alpha_1+\alpha_3-1}dy_1$$

The binomial series expansion of $\left[1-e^{(e^{2y_1}-1)}\right]^{\alpha_1+\alpha_3-1}$ is given by

$$\left[1-e^{(e^{2y_1}-1)}\right]^{\alpha_1+\alpha_3-1} = \sum_{j=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{j}(-1)^{-j}e^{-j(e^{2y_1}-1)}.$$

Then

$M_y(t) =$

$$2(\alpha_1+\alpha_3)\sum_{j=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{j}(-1)^{-j}\int_0^{\infty} e^{-ty_1}e^{2y}e^{-(e^{2y_1}-1)}\,dy_1. \quad (14)$$

$$= 2(\alpha_1+\alpha_3)\sum_{j=0}^{\infty}\binom{\alpha_1+\alpha_3-1}{j}(-1)^{-j}e^{(j+1)}\times$$

$$\int_0^{\infty} e^{-ty_1}e^{2y}e^{-e^{2y_1}(j+1)}\,dy_1$$

In addition, Taylor series expansion of $e^{e^{2y_1}(j+1)}$ is given by

$$e^{e^{2y_1}(j+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^k e^{2y_1 k}}{k!}.$$

Therefore, we can write (14) as follows:

$$M_y(t) = 2(\alpha_1+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(j+1)^k e^{2y_1 k}}{k!}.\binom{\alpha_1+\alpha_3-1}{j}(-1)^{-j}$$

$$\times\int_0^{\infty} e^{-ty_1}e^{2y}e^{-(e^{2y_1}-1)}\,dy_1.$$

$$= 2(\alpha_1+\alpha_3)\sum_{k=0}^{\infty}\frac{(-1)^{k+j}(j+1)^k\binom{\alpha_1+\alpha_3-1}{j}e^{(j+1)}}{k!\,(t-2-2k)}.$$

Note that: The moment generating function $M_{y_1}(t)$ can be used, instead of the marginal pdf $f\,(y_1)$, to derive the marginal expectation of Y_1 as:

$$E(y_1) = -\frac{d}{dt}M_{y_1}(t)|_{t=0}$$

$$-\frac{d}{dt}M_{y_1}(t) = 2(\alpha_1+\alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+j}(j+1)^k}{k!}\binom{\alpha_1+\alpha_3-1}{j}e^{(j+1)}\left[\frac{1}{(t-2-2k)^2}\right],$$

$at\ t=0,\ we\ get$

$$E_{y_1)} = 2(\alpha_1 + \alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+j}(j+1)^k}{k!}\binom{\alpha_1+\alpha_3-1}{j}e^{(j+1)}\left[\frac{1}{(-2-2k)^2}\right].$$

Similarly, the second moment of Y_1 can be derived from $M_{y_1}(t)$ as its second derivative at t=0 and r-th moment of Y_1 can be obtained by differentiating $M_{y_1}(t)$ r times with respect to t and putting $t = 0$

$$E(y_1^r) = -\frac{d^r}{dt^r}M_{y_1}(t)|_{t=0}$$

$$= 2(\alpha_1 + \alpha_3)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k+j}(j+1)^k}{k!}\binom{\alpha_1+\alpha_3-1}{j}(-1)^j e^{(j+1)}\left[\frac{r!}{(-2-2k)^{r+1}}\right]$$

3.2 The joint moment generating function:

If (Y_1, Y_2) having BOGEMW $(\alpha_1, \alpha_2, \alpha_3, \beta, \theta, \gamma, \lambda)$ distributions, then the joint moment generating function of (Y_1, Y_2) is given by:

$$M(t_1,t_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{(4)(\alpha_1+\alpha_3)\alpha_2(-1)^{k+j+m+i}(j+1)^k(j+1)^m}{k!\,m!\,(t_1-2-2k)(t_2-2-2m)}\times$$

$$\binom{\alpha_1+\alpha_3-1}{j}\binom{\alpha_2-1}{i}e^{(j+i+2)}$$

$$-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{(4)(\alpha_1+\alpha_3)\alpha_2(-1)^{k+j+m+i}(j+1)^k(i+1)^m}{k!\,m!\,(t_1-2-2k)(t_1+t_2-4-2m-2k)}\times$$

$$\binom{\alpha_1+\alpha_3-1}{j}\binom{\alpha_2-1}{i}e^{(j+i+2)}$$

$$-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{(4)(\alpha_1+\alpha_3)\alpha_2(-1)^{k+m+i+j}(j+1)^k(i+1)^m}{(t_1-2-2m)(t_2-2-2k)k!\,m!}\times$$

$$\binom{\alpha_2+\alpha_3-1}{j}\binom{\alpha_1-1}{i}e^{(j+i+2)}$$

$$-\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{(4)\alpha_1(\alpha_2+\alpha_3)(-1)^{k+m+i+j}(j+1)^k(i+1)^m}{(t_1+t_2-4-2k-2m)(t_2-2-2k)k!\,m!}\times$$

$$\binom{\alpha_2+\alpha_3-1}{j}\binom{\alpha_1-1}{i}e^{(j+i+2)}$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_1+\alpha_2+\alpha_3-1}{j}\frac{(2)\alpha_3(-1)^{k+j}(j+1)^k}{(t_1+t_2-2-2k)k!}e^{(j+1)}$$

4. Reliability Analysis

In this section we introduced some reliability measures such as the joint reliability function, joint hazard rate, joint mean waiting time, joint reversed (hazard) function and its marginal function.

4.1 Joint reliability function

Assume that (Y_1, Y_2) are two dimensional random variables with CDF $F_{Y_1,Y_2}(y_1, y_2)$, and the marginal function are $F_{Y_1}(y_1)$ and $F_{Y_2}(y_2)$ then, the joint reliability function $R_{Y_1,Y_2}(y_1, y_2)$ is

$$(15)\,R_{Y_1,Y_2}(y_1,y_2) = 1 - F_{Y_1}(y_1) - F_{Y_2}(y_2) + F_{Y_1,Y_2}(y_1,y_2).$$

Assume the random vector (Y_1, Y_2) has the BOGEMW then, the joint reliability function of (Y_1, Y_2) is given by

$$(16)\qquad R_{Y_1,Y_2}(y_1,y_2) = \begin{cases} R_1(y_1,y_2) & if\ \ 0 < y_1 < y_2 \\ R_2(y_1,y_2) & if\ \ 0 < y_2 < y_1 \\ R_3(y_1,y_2) & if\ \ \ \ y_1 = y_2, \end{cases}$$

where

$$R_{Y_1,Y_2}(y_1,y_2) = 1 - F_{y_1}(y_1) - F_{y_2}(y_2) + F_{y_1,y_2}(y_1,y_2).$$

It is equal to

$R_1(y_1, y_2) =$

$$1-\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3}-\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\right]^{\alpha_2+\alpha_3}$$

$$+\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3}\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\right]^{\alpha_2}.$$

In addition,

$$R_2(y_1,y_2)\ =1-F_{y_1}(y_1)-F_{y_2}(y_2)+F_2(y_1,y_2),$$

It is equal to

$$R_2(y_1,y_2)\ =$$

$$1-\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_1+\alpha_3}-\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\right]^{\alpha_2+\alpha_3}$$

$$+\left[1-e^{-\lambda\left(e^{\theta y_1+\gamma y_1^{\beta}}-1\right)}\right]^{\alpha_1}\left[1-e^{-\lambda\left(e^{\theta y_2+\gamma y_2^{\beta}}-1\right)}\right]^{\alpha_2+\alpha_3}.$$

Furthermore

$$R_3(y_1,y_2)\ =1-F_{y_1}(y_1)-F_{y_2}(y_2)+F_0(y).$$

It has been found that

$$R_3(y_1,y_2)=1-\left[1-e^{-\lambda(e^{\theta y}+\gamma y^{B}-1)}\right]^{\alpha_1+\alpha_2}-\left[1-e^{-\lambda(e^{\theta y}+\gamma y^{B}-1)}\right]^{\alpha_1+\alpha_2}$$

$$+\left[1-e^{-\lambda(e^{\theta y}+\gamma y^{B}-1)}\right]^{\alpha_1+\alpha_2+\alpha_3}$$

4.2 The joint hazard rate function and its marginal functions

Assum (Y_1,Y_2) are two dimensional random variable with PDF $f_{Y_1,Y_2(y_1,y_2)}$, and reliability function $R_{Y_1,Y_2}(y_1,y_2)$. Basu (1971) defined the bivariate hazard rate function as:

$$h(y_1,y_2)=\frac{f_{y_1,y_2}(Y_1,Y_2)}{R_{y_1,y_2}(Y_1,Y_2)} \tag{17}$$

Moreover, the bivariate hazard rate function for the random vector (Y_1,Y_2) which has the BOGEMW is

$$h(y_1, y_2) = \begin{cases} h_1(y_1, y_2) & if \quad 0 < y_1 < y_2 \\ h_2(y_1, y_2) & if \quad 0 < y_2 < y_1 \\ h_3(y, y) & if \quad y_1 = y_2 \end{cases} \qquad (18)$$

Where

$$h_1(y_1, y_2) = \frac{f_1(y_1, y_2)}{R_1(y_1, y_2)}$$

Therefore, we have

$$h_2(y_1, y_2) = \lambda(\alpha_1 + \alpha_3)\left(\theta + \beta\gamma {y_1}^{\beta-1}\right)e^{\theta y_1 + \gamma y_1^\beta} e^{-\lambda(e^{\theta y_1 + \gamma y_1^\beta} - 1)}$$

$$\left[Q_{y_1}\right]^{\alpha_1+\alpha_{3-1}} \times \frac{\lambda\alpha_2\left(\theta + \beta\gamma {y_2}^{\beta-1}\right)e^{\theta y_1 + \gamma y_2^\beta} e^{-\lambda(e^{\theta y_2 + \gamma y_2^\beta} - 1)}\left[Q_{y_2}\right]^{\alpha_1 - 1}}{1 - \left[Q_{y_1}\right]^{\alpha_1+\alpha_3} - \left[Q_{y_2}\right]^{\alpha_1+\alpha_3} + \left[Q_{y_1}\right]^{\alpha_1+\alpha_3}\left[Q_{y_2}\right]^{\alpha_1}}$$

$$h_2(y_1, y_2) = \frac{f_2(y_1, y_2)}{R_2(y_1, y_2)},$$

$$= \alpha_1\lambda\left(\theta + \beta\gamma {y_1}^{\beta-1}\right)e^{-\lambda(e^{\theta y_1 + \gamma y_1^\beta} - 1)} e^{\theta y_1 + \gamma y_1^\beta}\left[Q_{y_1}\right]^{\alpha_1+\alpha_{3-1}} \times$$

$$\frac{(\alpha_2+\alpha_3)\lambda\left(\theta+\beta\gamma {y_2}^{B-1}\right)e^{\theta y_2 + \gamma y_2^\beta} e^{-\lambda(e^{\theta y_2 + \gamma_2^\beta} - 1)}\left[Q_{y_2}\right]^{\alpha_2+\alpha_3-1}}{1 - \left[Q_{y_1}\right]^{\alpha_1+\alpha_3} - \left[Q_{y_2}\right]^{\alpha_1+\alpha_3} + \left[Q_{y1}\right]^{\alpha_1}\left[Q_{y_2}\right]^{\alpha_1+\alpha_3}},$$

and

$$h_3(y, y) = \frac{f_0(y)}{R_3(y, y)}$$

$$= \frac{\alpha_3\lambda(\theta + \beta\gamma y^{B-1})e^{\theta y + \gamma y^\beta} e^{-\lambda(e^{\theta y + \gamma y^\beta} - 1)}\left[Q_y\right]^{\alpha_1+\alpha_2+\alpha_3-1}}{1 - \left[Q_y\right]^{\alpha_1+\alpha_3} - \left[Q_y\right]^{\alpha_2+\alpha_3} + \left[Q_y\right]^{\alpha_1+\alpha_2+\alpha_3}}$$

Moreover, the marginal hazard rate functions $h(y_1)$ of the BOGEMW can be obtained from the marginal probability density functions of y_1 and the marginal reliability of y_1

$$h_{Y_1}(y_1) = \frac{\lambda(\alpha_1+\alpha_3)(\theta+\beta\gamma y_1{}^{\beta-1})e^{\theta y_1+\gamma y_1^{\beta}}e^{-\lambda\left(e^{\theta y1+\gamma y_1^{\beta}-1}\right)}[Q_{y_1}]^{\alpha_1+\alpha_3-1}}{1-[Q_{y_1}]^{\alpha_1+\alpha_3}}$$

Similarly

$$h_{Y_2}(y_2) = \frac{\lambda(\alpha_2+\alpha_3)(\theta+\beta\gamma y_2{}^{\beta-1})e^{\theta y_2+\gamma y_2^{\beta}}e^{-\lambda\left(e^{\theta y2+\gamma y_2^{\beta}-1}\right)}[Q_{y_2}]^{\alpha_2+\alpha_3-1}}{1-[Q_{y_2}]^{\alpha_2+\alpha_3}}$$

Where

$$Q_i = \left[1-e^{-\lambda\left(e^{\theta i+\gamma i^{B}-1}\right)}\right], \qquad i = y_1, y_2$$

4.3. The joint mean waiting time and its marginal functions

The waiting time is closely related to another important random variable reversed hazard rate function. Indeed as a condition of a failure in (0, t) is already imposed while defining the reversed hazard function, it is of interest in different applications (actuarial science, reliability analysis) to describe the time, which had elapsed since the failure. The most important applications of the waiting time are to describe different maintenance strategies to any system. The observations of waiting times can be used for predicting the distribution function. The joint mean waiting time function $M_w(t_1, t_2)$ is defined as follows.

$$M_w\,(t_1,t_2) = \frac{1}{F(t_1,t_2)}\int_0^{t_1}\int_0^{t_2} F(y_1,y_2)dy_2dy_1 \tag{19}$$

The following lemma obtains the joint mean waiting time of (y_1, y_2)

Lemma 2. The joint mean waiting time $M_w\,(t_1,t_2)$ to the random variables Y_1 and Y_2 is

$$M_w\,(t_1,t_2) = \begin{cases} M_{w_1}\,(t_1,t_2) & if\ t_1 < t_2 \\ M_{w_2}\,(t_1,t_2) & if\ t_1 > t_2 \\ M_{w_3}\,(t\,,t) & if\ t_1 = t_2 = t\ . \end{cases} \tag{20}$$

Where

$$M_{w1}\,(t_1,t_2)=\frac{1}{F(t_1,t_2)}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{d=0}^{\infty}\binom{j}{k}^2\binom{\alpha_2}{m}\binom{\alpha_2+\alpha_3}{m}\frac{(\lambda m)^{2j}(\gamma(j-k)^{2d}}{(j!\,d!)^2(\theta(j-k))^{2(\beta d+1)}}\times$$

$$e^{\theta(j-k)t_1}\left[(\theta(j-k)t_1)^{\beta d}+\sum_{i=1}^{\beta d}(-1)^i\,i!\binom{\beta d}{i}(\theta(j-k)t_1)^{\beta d-i}\right]\times$$

$$e^{\theta(j-k)t_2}\left[(\theta(j-k)t_2)^{\beta d}+\sum_{i=1}^{\beta d}(-1)^i\,i!\binom{\beta d}{i}(\theta(j-k)t_2)^{\beta d-i}\right]$$

$$M_{w2}\,(t_1,t_2)=\frac{1}{F(t_1,t_2)}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{d=0}^{\infty}\binom{j}{k}^2\binom{\alpha_2}{m}\binom{\alpha_2+\alpha_3}{m}\frac{(\lambda m)^{2j}(\gamma(j-k)^{2d}}{(j!\,d!)^2(\theta(j-k))^{2(\beta d+1)}}\times$$

$$e^{\theta(j-k)t_1}\left[(\theta(j-k)t_1)^{\beta d}+\sum_{i=1}^{\beta d}(-1)^i\,i!\binom{\beta d}{i}(\theta(j-k)t_1)^{\beta d-i}\right]\times$$

$$e^{\theta(j-k)t_2}\left[(\theta(j-k)t_2)^{\beta d}+\sum_{i=1}^{\beta d}(-1)^i\,i!\binom{\beta d}{i}(\theta(j-k)t_2)^{\beta d-i}\right]$$

and

$$M_{\omega_3}(t\,,t)$$
$$=\frac{1}{F(t\,,t)}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{d=0}^{\infty}\binom{j}{k}\binom{\alpha_1+\alpha_2+\alpha_3}{m}\frac{(-1)^{j+k+m}(\lambda m)^j\big(\gamma(j-k)\big)^d}{j\,!d\,!\big(\theta(j-k)\big)^{\beta d+1}}$$
$$\times\,e^{\theta(j-k)t}\left[(\theta(j-k)t)^{\beta d}\;+\sum_{i=1}^{\beta d}(-1)^i\;i!\binom{\beta d}{i}(\theta(j-k)t)^{\beta d-i}\right].$$

4.4 The joint reversed hazard rate function and its marginal functions

Assume (Y_1,Y_2) are two dimensional random variable with CDF $F_{y_1,y_2}(y_1,y_2)$ and pdf $f_{Y_1,Y_2}(y_1,y_2)$. Then joint reversed hazard rate function is

$$r\ (y_1, y_2) = \frac{f_{y_1,y_2}(y_1,y_2)}{F_{y_1,y_2}(y_1,y_2)}. \tag{21}$$

Thus, the bivariate reversed hazard rate function for the random vector (y_1, y_2) which has the BOGEMW is

$$r_{Y_1,Y_2}(y_1, y_2) = \begin{cases} r_1(y_1, y_2) & if \quad 0 < y_1 < y_2 \\ r_2(y_1, y_2) & if \quad 0 < y_2 < y_1 \\ r_3(y, y) & if \quad y_1 = y_2 = y \end{cases}.$$

(22)

When

$$r\ (y_1, y_2) = \frac{f_1(y_1, y_2)}{F_1(y_1, y_2)}$$

then, we have

$$r_1(y_1, y_2) = \lambda(\alpha_1 + \alpha_3)\left(\theta + \beta\gamma y_1^{\beta-1}\right) e^{\theta y_1 + \gamma y_1^{\beta-1}} e^{-\lambda\left(e^{\theta y_1 + \gamma y_1^{\beta}} - 1\right)} \times$$

$$\frac{\lambda\alpha_2\left(\theta + \beta\gamma y_2^{\beta-1}\right) e^{\theta y_2 + \gamma y_2^{\beta}}\ e^{-\lambda\left(e^{\theta y_2 + \gamma y_2^{\beta}} - 1\right)}}{\left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma y_1^{\beta}} - 1\right)}\right]\left[1 - e^{-\lambda\left(e^{\theta y_2 + \gamma y_2^{\beta}} - 1\right)}\right]},$$

while

$$r_2(y_1, y_2) = \frac{f_2(y_1, y_2)}{F_2(y_1, y_2)}$$

therefore, it is equal to

$$r_2(y_1, y_2) = \alpha_1\lambda\left(\theta + \beta\gamma y_1^{\beta-1}\right) e^{\theta y_1 + \gamma y_1^{\beta}} e^{-\lambda\left(e^{\theta y_1 + \gamma y_1^{\beta}} - 1\right)} \times$$

$$\frac{\lambda(\alpha_2 + \alpha_3)\left(\theta + \beta\gamma y_2^{\beta-1}\right) e^{\theta y_2 + \gamma y_2^{\beta}}\ e^{-\lambda\left(e^{\theta y_2 + \gamma y_2^{\beta}} - 1\right)}}{\left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma y_1^{\beta}} - 1\right)}\right]\left[1 - e^{-\lambda\left(e^{\theta y_2 + \gamma y_2^{\beta}} - 1\right)}\right]}.$$

Also,

$$r_3\ (y, y) = \frac{f_0(y)}{F_0(y)},$$

hence

$$r_3(y, y) = \frac{\alpha_3 \lambda\left(\theta + \beta\gamma y^{\beta-1}\right) e^{\theta y + \gamma y^{\beta}} e^{-\lambda\left(e^{\theta y + \lambda y^{\beta}} - 1\right)}}{\left[1 - e^{-\lambda\left(e^{\theta y + \gamma y^{\beta}} - 1\right)}\right]}.$$

In addition, the marginal reversed hazard rate function $r_{Y_1}(y_1)$ and $r_{Y_2}(y_2)$ to the BOGEMW are

$$r_{y_1}(y_1) = \frac{f_{y_1}(y_1)}{f_{y_1}(y_1)}.$$

It is equal to

$$r_{Y_1}(y_1) = \frac{\lambda(\alpha_1 + \alpha_3)\left(\theta + \beta\gamma_{y_1}^{\beta-1}\right) e^{\theta y_1 + \gamma y_1^{\beta-1}} e^{-\lambda\left(e^{\theta y_1 + \lambda y_1^{\beta}} - 1\right)}}{\left[1 - e^{-\lambda\left(e^{\theta y_1 + \gamma y_1^{\beta}} - 1\right)}\right]}.$$

$$r_{Y_2}(y_2) = \frac{\lambda(\alpha_2 + \alpha_3)\left(\theta + \beta\gamma y_2^{\beta-1}\right) e^{\theta y_2 + \gamma y_2^{\beta-1}} e^{-\lambda\left(e^{\theta y_2 + \lambda y_2^{\beta}} - 1\right)}}{\left[1 - e^{-\lambda\left(e^{\theta y_2 + \gamma y_1^{\beta}} - 1\right)}\right]}$$

5. Maximum Likelihood Estimation

In this section we used maximum likelihood to estimate the unknown parameters of the BOGEMW distribution. We use the same argument as that given in Kundu and Gupta (2009). We want to estimate the other parameters$(\alpha_1, \alpha_2, \alpha_3, \beta, \theta, \gamma, \lambda)$. Suppose that we have a sample of size n, of the form$\{(y_1,y_2),\dots\ (y_{1n},y_{2n})\}$ from BOGEMW distribution. We use the following notation

$n_1=(i;y_{1i}<y_{2i})$, $n_2=(i;y_{1i}>y_{2i})$, $n_3=(i;y_{1i}=y_{2i})$ where $n=n_1+n_2+n_3$.

Based on the observations, and using the density functions $f_1(y_1,y_2), f_2(y_1,y_2)$ and $f_0(y)$ the likelihood function is given by

$$l(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma,\lambda)=\prod_{i=1}^{n_1} f_1(y_{1i},y_{2i})\prod_{i=1}^{n_2} f_2(y_{1i},y_{2i})\prod_{i=1}^{n_3} f_0(y_i)$$

The log-likelihood function can be written as

$$l(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma,\lambda)$$
$$= n_1\ln\alpha_2+n_1\ln(\alpha_1+\alpha_3)+n_2\ln\alpha_1$$
$$+n_2\ln(\alpha_2+\alpha_3)+n_3\ln\alpha_3$$

$$+(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\ln\left[1-e^{-\lambda\left(e^{\theta y_{1i}+\gamma y_{1i}{}^{\beta}}-1\right)}\right]+\ (\alpha_2-1)\times$$

$$\sum_{i=1}^{n_1}\ln\left[1-e^{-\lambda\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)}\right]+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}\ln\left[1-e^{-\lambda\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)}\right]$$

$$+(\alpha_2-1)\sum_{i=1}^{n_2}\ln\left[1-e^{-\lambda\left(e^{\theta y_{i1}+\lambda y_{i1}^{\beta}}-1\right)}\right]+\sum_{i=1}^{n_1}\left(\theta y_{1i}+\gamma y_{1i}^{\beta}\right)+$$

$$(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\ln\left[1-e^{-\lambda\left(e^{\theta y_i+\lambda y_i^{\beta}}-1\right)}\right]-\lambda\sum_{i=1}^{n_1}\left(e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}-1\right)$$

$$+\sum_{i=1}^{n_1}\left(\theta y_{2i}+\gamma y_{2i}^{\beta}\right)-\lambda\sum_{i=1}^{n_1}\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)+\sum_{i=1}^{n_2}\left(\theta y_{1i}+\gamma y_{1i}^{\beta}\right)$$

$$-\lambda\sum_{i=1}^{n_2}\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)-\lambda\sum_{i=1}^{n_2}\left(e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}-1\right)+\sum_{i=1}^{n_2}\left(\theta y_{2i}+\gamma y_{2i}^{\beta}\right)$$

$$\sum_{i=1}^{n_1}\ln\left(\theta+\beta\gamma y_{2i}^{\beta-1}\right)+\sum_{i=1}^{n_2}\left(\theta+\beta\gamma y_{1i}^{\beta-1}\right)+\sum_{i=1}^{n_2}\ln\left(\theta+\beta\gamma y_{1i}^{\beta-1}\right)+$$

$\sum_{i=1}^{n_3}\left(\theta y_i+\gamma y_i^{\beta}\right)-\lambda\sum_{i=1}^{n_3}\left(e^{\theta y_i+\gamma y_i^{\beta}}-1\right)++\sum_{i=1}^{n_2}\ln\left(\theta+\beta\gamma y_{2i}^{\beta-1}\right)+$

(23)$\sum_{i=1}^{n_3}\ln\left(\theta+\beta\gamma y_i^{\beta-1}\right)+(2(n_1+n_2)+n_3)\ln\lambda$

Computing the first partial derivatives of (23) with respect to $\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma\ and\ \lambda$ and setting the results equal zeros, we get the likelihood equations as in the following form

$$\frac{\partial l}{\partial\alpha_1}=\frac{n_1}{\alpha_1+\alpha_3}+\frac{n_2}{\alpha_1}+\sum_{i=1}^{n_1}\ln\left[1-e^{-\lambda\left(e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}-1\right)}\right]+$$

$$\sum_{i=1}^{n_2}\ln\left[1-e^{-\lambda\left(e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}-1\right)}\right]+\sum_{i=1}^{n_3}\ln\left[1-e^{-\lambda\left(e^{\theta y_i+\gamma y_i^{\beta}}-1\right)}\right]$$

$$\frac{\partial l}{\partial\alpha_2}=\frac{n_2}{\alpha_2+\alpha_3}+\frac{n_1}{\alpha_2}+\sum_{i=1}^{n_1}\ln\left[1-e^{-\lambda\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)}\right]$$

$$+\sum_{i=1}^{n_2}\ln\left[1-e^{-\lambda\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)}\right]+\sum_{i=1}^{n_3}\ln\left[1-e^{-\lambda\left(e^{\theta y_i+\gamma y_i^{\beta}}-1\right)}\right]$$

$$\frac{\partial l}{\partial\alpha_3}=\frac{n_1}{\alpha_1+\alpha_3}+\frac{n_2}{\alpha_2+\alpha_3}+\frac{n_3}{\alpha_3}+\sum_{i=1}^{n_1}\ln\left[1-e^{-\lambda\left(e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}-1\right)}\right]$$

$$+\sum_{i=1}^{n_2}\ln\left[1-e^{-\lambda\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)}\right]+\sum_{i=1}^{n_3}\ln\left[1-e^{-\lambda\left(e^{\theta y_i+\gamma y_i^{\beta}}-1\right)}\right]$$

and

$$\frac{\partial l}{\partial\lambda}=(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\left[\frac{\xi\left(y_{1i},\underline{\Theta_1}\right)-1}{\vartheta\left(y_{1i},\underline{\Theta_2}\right)}\right]+(\alpha_2-1)\sum_{i=1}^{n_1}\left[\frac{\xi\left(y_{2i},\underline{\Theta_1}\right)-1}{\vartheta\left(y_{2i},\underline{\Theta_2}\right)}\right]$$
$$+(\alpha_2+\alpha_3-1)$$

$$\times\sum_{i=1}^{n_2}\left[\frac{\xi\left(y_{2i},\underline{\Theta_1}\right)-1}{\vartheta\left(y_{2i},\underline{\Theta_2}\right)}\right]+(\alpha_1-1)\sum_{i=1}^{n_2}\left[\frac{\xi\left(y_{1i},\underline{\Theta_1}\right)-1}{\vartheta\left(y_{1i},\underline{\Theta_2}\right)}\right]+(\alpha_1+\alpha_2+\alpha_3-1)$$

$$\times\sum_{i=1}^{n_3}\left[\frac{\xi\left(y_i,\underline{\Theta_1}\right)-1}{\vartheta\left(y_i,\underline{\Theta_2}\right)}\right]-\sum_{i=1}^{n_1}\left(\xi\left(y_{1i},\underline{\Theta_1}\right)-1\right)-\sum_{i=1}^{n_1}\left(\xi\left(y_{2i},\underline{\Theta_1}\right)-1\right)$$

$$-\sum_{i=1}^{n_2}\left(\xi\left(y_{1i},\underline{\Theta_1}\right)-1\right)-\sum_{i=1}^{n_2}\left(\xi\left(y_{2i},\underline{\Theta_1}\right)-1\right)-\sum_{i=1}^{n_3}\left(\xi\left(y_i,\underline{\Theta_1}\right)-1\right)$$
$$+\frac{(2(n_1+n_2)+n_3)}{\lambda}$$

Where $\underline{\Theta_1}=(\beta,\theta,\gamma)$ and the following nonlinear functions are given by

$$,\ \xi\left(y_{2i},\underline{\Theta_1}\right)=e^{\theta y_{2i}+\gamma y_{1i}^{\beta}},\ \xi\left(y_i,\underline{\Theta_1}\right)=e^{\theta y_i+\gamma y_i^{\beta}},\ \xi\left(y_{1i},\underline{\Theta_1}\right)=e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}$$

$$\vartheta\left(y_{1i},\underline{\Theta_2}\right)=e^{-\lambda\left(e^{\theta y_{1i}+\gamma y_{1i}^{\beta}}-1\right)}-1,\qquad \vartheta\left(y_{2i},\underline{\Theta_2}\right)=e^{-\lambda\left(e^{\theta y_{2i}+\gamma y_{2i}^{\beta}}-1\right)}-1$$

and $\vartheta\left(y_i,\underline{\Theta_2}\right)=e^{-\lambda\left(e^{\theta y_i+\gamma y_i^{\beta}}-1\right)}-1,$

$$\frac{\partial l}{\partial\gamma}=(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\left[\lambda y_{1i}^{\beta}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+(\alpha_2-1)$$

$$\times\sum_{i=1}^{n_1}\left[\lambda y_{2i}^{\beta}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}\left[\lambda y_{2i}^{\beta}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]$$

$$+(\alpha_1-1)\sum_{i=1}^{n_1}\left[\lambda y_{1i}^{\beta}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+\sum_{i=1}^{n_1}y_{1i}^{\beta}+(\alpha_1+\alpha_2+\alpha_3-1)$$

$$\times\sum_{i=1}^{n_3}\left[\lambda y_i^{\beta}\Omega\left(y_i,\underline{\Theta_2}\right)\right]-\lambda\sum_{i=1}^{n_1}y_{1i}^{\beta}\xi\left(y_{1i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_1}y_{2i}^{\beta}+\sum_{i=1}^{n_2}y_{1i}^{\beta}$$

$$-\lambda\sum_{i=1}^{n_1}y_{2i}^{\beta}\xi\left(y_{2i},\underline{\Theta_1}\right)-\lambda\sum_{i=1}^{n_2}y_{2i}^{\beta}\xi\left(y_{1i},\underline{\Theta_1}\right)-\lambda\sum_{i=1}^{n_2}y_{2i}^{\beta}\xi\left(y_{2i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_2}y_{2i}^{\beta}$$

$$-\lambda\sum_{i=1}^{n_3}{y_i}^{\beta}\xi\left(y_i,\underline{\Theta_1}\right)+\sum_{i=1}^{n_1}\frac{\beta y_{1i}^{\beta-1}}{\left(\theta+\beta\gamma y_{1i}^{\beta-1}\right)}+\sum_{i=1}^{n_1}\frac{\beta y_{2i}^{\beta-1}}{\left(\theta+\beta\gamma y_{2i}^{\beta-1}\right)}$$

$$,+\sum_{i=1}^{n_2}\frac{\beta y_{1i}^{\beta-1}}{\left(\theta+\beta\gamma y_{1i}^{\beta-1}\right)}+\sum_{i=1}^{n_2}\frac{\beta y_{2i}^{\beta-1}}{\left(\theta+\beta\gamma y_{2i}^{\beta-1}\right)}+\sum_{i=1}^{n_3}y_i^{\beta}$$

where

$$\Omega\left(y_{1i},\underline{\Theta_2}\right)=\frac{\xi\left(y_{1i},\underline{\Theta_1}\right)}{\vartheta\left(y_{1i},\underline{\Theta_2}\right)},\Omega\left(y_{2i},\underline{\Theta_2}\right)=\frac{\xi\left(y_{2i},\underline{\Theta_1}\right)}{\vartheta\left(y_{2i},\underline{\Theta_2}\right)}\ and\ \Omega\left(y_i,\underline{\Theta_2}\right)=\frac{\xi\left(y_i,\underline{\Theta_1}\right)}{\vartheta\left(y_i,\underline{\Theta_2}\right)}$$

$$\frac{\partial l}{\partial\theta}=(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\left[\lambda y_{1i}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+(\alpha_2-1)\sum_{i=1}^{n_1}\left[\lambda y_{2i}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]$$

$$+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2}\left[\lambda y_{2i}\Omega\left(y_{2i},\underline{\Theta_2}\right)\right]+(\alpha_1-1)\sum_{i=1}^{n_2}\left[\lambda y_{1i}\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]$$

$$+(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\left[\lambda y_i\Omega\left(y_i,\underline{\Theta_2}\right)\right]+\sum_{i=1}^{n_1}y_{1i}-\lambda\sum_{i=1}^{n_1}y_{1i}\xi\left(y_{1i},\underline{\Theta_1}\right)$$

$$+\sum_{i=1}^{n_1}y_{2i}-\lambda\sum_{i=1}^{n_1}y_{2i}\xi\left(y_{2i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_2}y_{1i}-\lambda\sum_{i=1}^{n_2}y_{1i}\xi\left(y_{1i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_2}y_{2i}$$

$$-\lambda\sum_{i=1}^{n_2}y_{2i}\xi\left(y_{2i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_1}\frac{1}{\left(\theta+\beta\gamma y_{1i}^{\beta-1}\right)}+\sum_{i=1}^{n_1}\frac{1}{\left(\theta+\beta\gamma y_{2i}^{\beta-1}\right)}$$

$$+\sum_{i=1}^{n_2}\frac{\beta y_{1i}^{\beta-1}}{\left(\theta+\beta\gamma y_{1i}^{\beta-1}\right)}+\sum_{i=1}^{n_2}\frac{\beta y_{2i}^{\beta-1}}{\left(\theta+\beta\gamma y_{2i}^{\beta-1}\right)}-\lambda\sum_{i=1}^{n_3}y_i\xi\left(y_i,\underline{\Theta_1}\right)+\sum_{i=1}^{n_3}y_i^{\beta}$$

$$\frac{\partial l}{\partial\beta}=(\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1}\left[\lambda y_{1i}^{\beta}\ln y_{1i}\,\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+(\alpha_2-1)$$

$$\sum_{i=1}^{n_1}\left[\lambda y_{2i}^{\beta}\ln y_{2i}\,\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+(\alpha_2+\alpha_3-1)\sum_{i=1}^{n_1}\left[\lambda y_{2i}^{\beta}\ln y_{2i}\,\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]$$

$$+(\alpha_1-1)\sum_{i=1}^{n_2}\left[\lambda y_{1i}^{\beta}\ln y_{1i}\,\Omega\left(y_{1i},\underline{\Theta_2}\right)\right]+\sum_{i=1}^{n_1}y_{1i}^{\beta}\gamma\ln y_{1i}+$$

$$(\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3}\left[\lambda y_i^{\beta}\ln y_{1i}\,\Omega\left(y_i,\underline{\Theta_2}\right)\right]$$

$$-\lambda\gamma\sum_{i=1}^{n_1}y_{1i}^{\beta}\ln y_{1i}\,\xi\left(y_{1i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_1}y_{2i}^{\beta}\gamma\ln y_{2i}+\sum_{i=1}^{n_2}y_{1i}^{\beta}\,\gamma\ln y_{1i}$$

$$-\lambda\gamma\sum_{i=1}^{n_2}y_{1i}^{\beta}\ln y_{1i}\,\xi\left(y_{1i},\underline{\Theta_1}\right)-\lambda\gamma\sum_{i=1}^{n_1}y_{2i}^{\beta}\ln y_{2i}\,\xi\left(y_{2i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_3}y_i^{\beta}\gamma\ln y_i$$

$$-\lambda\gamma\sum_{i=1}^{n_2}y_{2i}^{\beta}\ln y_{2i}\,\xi\left(y_{2i},\underline{\Theta_1}\right)+\sum_{i=1}^{n_2}y_{2i}^{\beta}\gamma\ln y_{2i}-\lambda\gamma\sum_{i=1}^{n_3}y_i{}^{\beta}\ln y_i\,\xi\left(y_i,\underline{\Theta_1}\right)$$

$$+\sum_{i=1}^{n_1}\left[\frac{\beta y_{1i}^{\beta-1}+\beta\gamma y_{1i}^{\beta-1}\ln y_{1i}}{\theta+\beta\gamma y_{1i}^{\beta-1}}\right]+\sum_{i=1}^{n_1}\left[\frac{\beta y_{2i}^{\beta-1}+\beta\gamma y_{2i}^{\beta-1}\ln y_{2i}}{\theta+\beta\gamma y_{2i}^{\beta-1}}\right]$$

$$.+\sum_{i=1}^{n_2}\left[\frac{\beta y_{1i}^{\beta-1}+\beta\gamma y_{1i}^{\beta-1}\ln y_{1i}}{\theta+\beta\gamma y_{1i}^{\beta-1}}\right]+\sum_{i=1}^{n_2}\left[\frac{\beta y_{2i}^{\beta-1}+\beta\gamma y_{2i}^{\beta-1}\ln y_{2i}}{\theta+\beta\gamma y_{2i}^{\beta-1}}\right]$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero $\mathrm{I}\left(\underline{\hat{\varphi}}\right)=0$ where $\underline{\varphi}=(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma,\lambda)$. Solving these equations simultaneously gives the MLEs $\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\beta},\hat{\theta},\hat{\gamma}$ *and* $\hat{\lambda}$. For the BOGEMW distribution all the second order derivatives exist. The interval estimation of the model parameters requires the 7×7 observed information matrix $\mathrm{I}\left(\underline{\hat{\varphi}}\right)=\{I_{ij}\}$ for $i,j=\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma$ *and* λ. The multivariate normal $N_7\left(0,\mathrm{I}\left(\underline{\hat{\varphi}}\right)^{-1}\right)$ distribution, under standard regularity conditions, can be used to provide approximate confidence intervals for the unknown parameters, where $\mathrm{I}\left(\underline{\hat{\varphi}}\right)$ is the total observed information matrix evaluated at $\underline{\varphi}$. Then, approximate $100(1-\delta)\%$ confidence intervals for $\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma$ *and* λ can be determined by: $\alpha_1 \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\alpha_1\alpha_1}}$ $\alpha_2 \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\alpha_2\alpha_2}}$, $\alpha_3 \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\alpha_3\alpha_3}}$, $\beta \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\beta\beta}}$, $\theta \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\theta\theta}}$, $\gamma \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\gamma\gamma}}$ and $\lambda \pm z_{\frac{\delta}{2}}\sqrt{\hat{I}_{\lambda\lambda}}$ where $z_{\frac{\delta}{2}}$ is the upper δth percentile of the standard normal model.

6. Simulation study

In this section, we assess the performance of the MLEs of the BOGEMW parameters using Monte Carlo simulations. For different combinations $\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma$ *and* λ of samples of sizes n = 20, 50, 100 and 200 are generated from the BOGEMW model. The empirical results are given in Table 1. It is evident that the estimates are quite stable and close to the true values of the parameters for these sample sizes. Additionally, as the sample size increases, the biases and the standard errors of the MLEs decrease as expected. This study presents an assessment of the properties for MLE in terms of bias and mean square error (MSE) as well as the BCI for the parameters. The following algorithm shows how to generate data from the BOGEMW distribution.

- Generate A_1, A_2 *and* A_3 from $A(0,1)$
- Compute $U_i=\left(-\frac{-\ln\left(\left((-1/\lambda)\ln\left(1-A^{1/\alpha}\right)+1\right)+\theta A\right)}{\gamma}\right)$; $i=1,2,3$

- Obtain $Y_1 = \max\{U_1, U_3\}$ and $Y_2 = \max\{U_2, U_3\}$.

Table (1): Estimation summaries for the BOGEMW distribution based on complete data.

Parameters	n=20		n=50		n=100		n=200	
	Bais	MSE	Bais	MSE	Bais	MSE	Bais	MSE
θ=0.1	-0.055	.0048	-0.057	0.004	-0.053	0.003	-0.053	0.003
γ=0.1	-1.28	1.781	-0.579	0.411	-0.364	0.15	-0.303	0.096
λ=1.3	0.331	0.19	0.246	0.071	0.258	0.068	0.263	0.069
α_1=0.5	0.476	0.229	0.449	0.204	0.442	0.199	0.436	0.191
α_2=0.6	0.513	0.265	0.426	0.225	0.39	0.154	0.371	0.139
α_3=0.7	0.571	0.331	0.449	0.267	0.392	0.158	0.369	0.138
θ=0.2	-0.09	0.018	-0.045	0.009	-0.022	0.002	-0.021	0.002
γ=0.2	0.116	0.016	0.092	0.011	0.084	0.008	0.079	0.007
λ=1	-0.53	0.346	-0.496	0.319	-0.49	0.283	-0.461	0.244
α_1=0.4	0.05	0.084	0.012	0.049	0.068	0.036	0.071	0.016
α_2=0.5	-0.31	0.125	-0.304	0.12	-0.314	0.116	-.308	0.108
α_3=0.6	-0.55	0.362	-0.512	0.325	-0.507	0.295	-0.482	0.257
θ=0.3	0.108	0.446	-0.242	0.115	-0.146	0.051	-0.093	0.031
γ=0.3	0.585	0.394	0.408	0.189	0.375	0.148	0.36	0.135
λ=1.2	-1.35	2.167	-1.105	1.316	-1.275	0.904	-1.018	0.095
α_1=0.1	-0.062	0.134	-0.418	0.134	-0.484	0.301	-0.478	0.178
α_2=0.2	-0.465	0.304	-0.865	0.304	-0.988	0.201	-1.082	0.027
α_3=0.3	-1.298	1.827	-0.993	1.827	-0.824	1.316	-1.224	0.585

7. Data Analysis

In this section we present the analysis of a bivariate two real data sets to illustrate the performance of the BOGEMW distribution in practice. The data sets are used to illustrate that the BOGEMW distribution may fit better than bivariate generalized Gompertz (BGG) distribution and bivariate exponentiated generalized Weibull-Gompertz (BEGWG) distribution.

- ## The first real data

Here, consider the data obtained by Meintanis (2007), which represent football (soccer) data. This data describes the games where at least one kick goal scored by any team has been considered, and the home team must have scored at least one goal. Figure (2) shows that the scatter plot for the first data set and The PDF and estimated CDF for Y_1, Y_2 and $\min(y_1, y_2\ )$ displayed in Figures 3-5 which support our results in Table (2) .

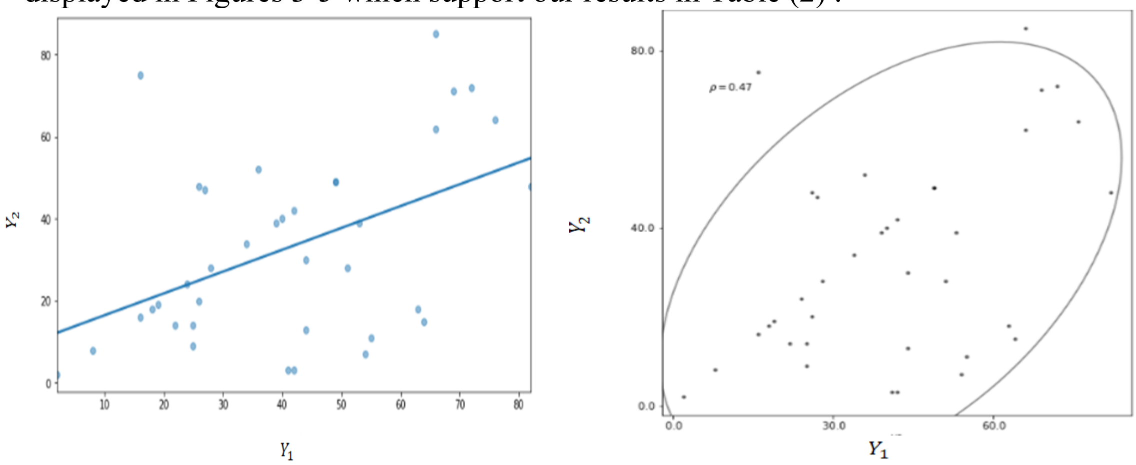


Figure (2): The scatter plot for first data set

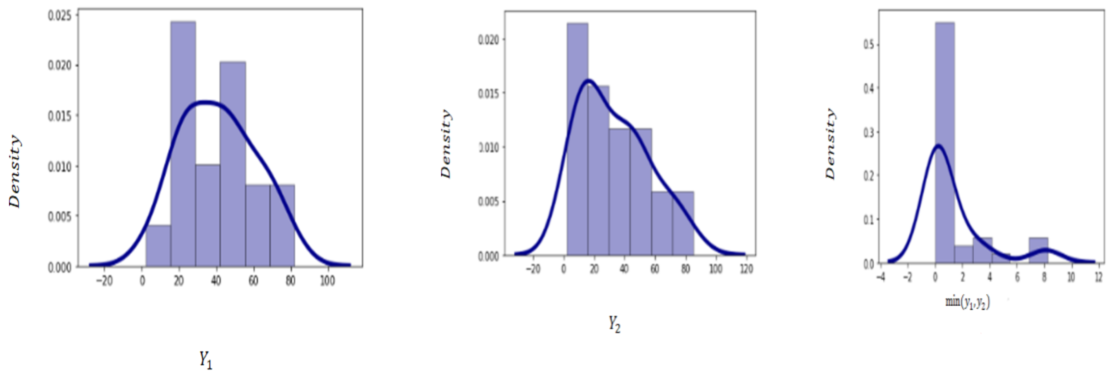
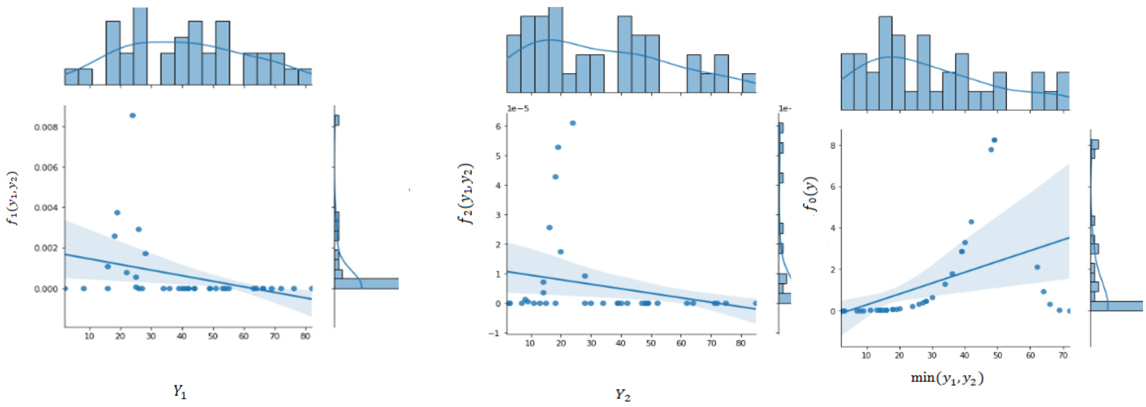


Figure (3): PDF for Y_1, Y_2 and $\min(y_1, y_2)$



Figure(4): Y_1, Y_2 and min(Y_1,Y_2) with $f_1(y_1, y_2), f_2(y_1, y_2)$ and $f_0(y)$ respectively for second data set

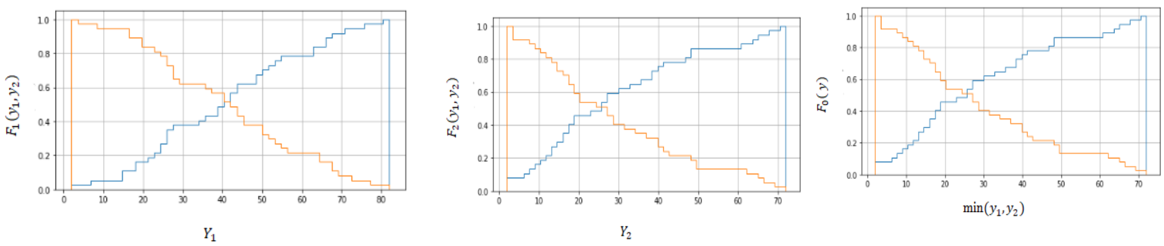


Figure (5): Y_1, Y_2 and $\min(Y_1, Y_2)$ with $F_1(y_1, y_2), F_2(y_1, y_2)$ and $F_0(y)$ respectively for first data set

In order to compare the distributions, we consider the following criteria: The l (maximized log-likelihood), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan Quinn information criterion (HQIC). Also, we apply formal goodness-of fit tests in order to verify which distribution fits better to these data. The model with minimum values for these statistics could be chosen as the best model to fit the data.

Table (2): *l*, AIC, CAIC, BIC and HQIC

The Model	l	AIC	CAIC	BIC	HQIC
BEGWG	-318.152	644.304	645.554	650.747	646.575
BGG	-327.663	663.327	664.577	669.771	665.599
BOGEMW	-216.689	441.378	442.628	447.822	443.65

Table (2) shows that BOGEMW distribution is the best distribution because it has the smallest value of AIC, CAIC, BIC and HQIC test.

- **The second real data**

The exchange rate plays an important role in the economic activities undertaken by countries, whether the activity is commercial or investment as the exchange rate occupies a central position in monetary policy because of its use as a goal or as a tool or as an indicator of international competitiveness through its impact on economic growth components such as investment and the degree of openness to foreign trade. The exchange rate is defined as the number of units of the local currency against one unit of foreign currency or vice versa and the exchange rate represents a link between commodity prices in the local economy and their prices in the global market. There are several concepts of the exchange rate for example, The trade-weighted effective exchange rate index, a common form of the effective exchange rate index, is a multilateral exchange rate index. It is compiled as a weighted average of exchange rates of home versus foreign currencies, with the weight for each foreign country equal to its share in trade. Depending on the purpose for which it is used, it can be export-weighted, import-weighted, or total-external trade weighted. The trade-weighted effective exchange rate index is an economic indicator for comparing the exchange rate of a country against those of their major trading partners. By design, movements in the currencies of those trading partners with a greater share in an economy's exports and imports will have a greater effect on the effective exchange rate. In a multilateral, highly globalized, world, the effective exchange rate index is much more useful than a bilateral exchange rate.

The data set represents The trade-weighted effective exchange rate for the Egyptian pound against the US dollar and the trade-weighted effective exchange rate the for Turkish lira against the US dollar for the year 2010 : 2013 are represented in Table (3). The data were published in Abo El yazzed (2016). It is a bivariate data set, and the variables y_1 and y_2 are as follows:

1. y_1 represents the trade-weighted effective exchange rate for the Egyptian pound against the US dollar using monthly data,
2. y_2 represents the trade-weighted effective exchange rate for the Turkish lira against the US dollar using monthly data.

Table (3): The Trade-weighted effective exchange rate for years 2010 : 2013. (Abo El yazzed (2016)).

Date	Y_1	Y_2	Date	Y_1	Y_2	Date	Y_1	Y_2
29/01/2010	101.944	98.3729	30/04/2011	88.228	90.972	31/07/2012	95.182	86.714
26/02/2010	102.252	96.0483	31/05/2011	89.782	88.024	31/08/2012	94.010	84.457
31/03/2010	101.939	97.8967	30/06/2011	89.035	86.609	30/09/2012	92.718	84.457
30/04/2010	101.255	101.255	30/07/2011	89.130	84.042	31/10/2012	92.606	83.815
31/05/2010	103.479	101.119	31/08/2011	89.466	81.159	30/11/2012	92.218	84.252
30/06/2010	102.798	100.046	30/09/2011	93.699	79.788	31/12/2012	88.158	83.324
30/07/2010	99.3829	100.348	31/10/2011	91.355	82.096	31/01/2013	82.732	83.486
31/08/2010	100.532	101.187	30/11/2011	92.995	80.746	28/02/2013	83.929	83.929
30/09/2010	96.955	101.441	30/12/2011	94.128	79.220	30/03/2013	83.994	84.171
29/10/2010	94.3443	101.851	31/01/2012	92.887	83.703	30/04/2013	81.325	84.098
30/11/2010	97.2018	100.211	29/02/2012	91.859	83.708	31/05/2013	81.972	81.972
31/12/2010	95.2085	95.2085	30/03/2012	92.290	83.301	30/09/2013	82.095	73.403

31/01/2011	93.7348	91.0215	30/04/2012	92.402	84.442	31/10/2013	81.782	74.215
28/02/2011	92.7524	91.1574	31/05/2012	96.337	84.203	30/11/2013	82.258	73.951
31/03/2011	90.3675	92.1948	29/06/2012	94.657	85.128	31/12/2013	81.316	69.737

Figure (6) shows that the scatter plot for the first data set and Figures (7-9) show PDF and estimated CDF for Y_1, Y_2 and $\min(Y_1, Y_2)$ which support our results in Table (4) .

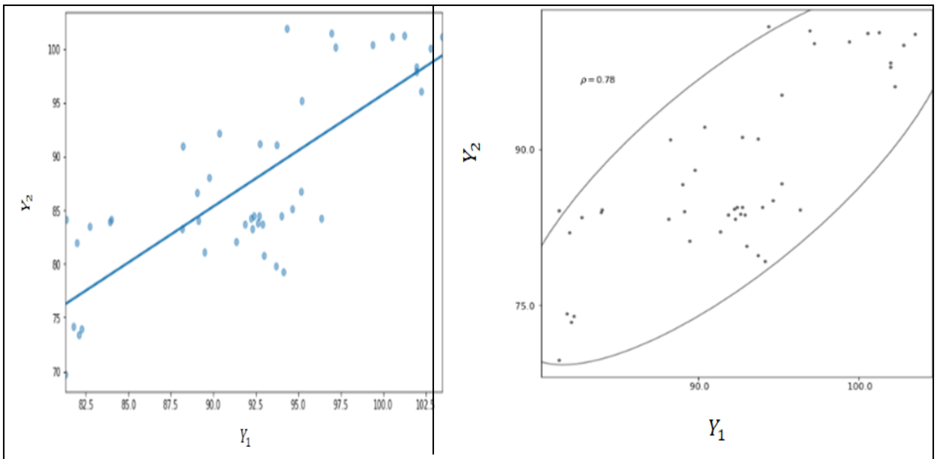


Figure (6): The scatter plot for second data set

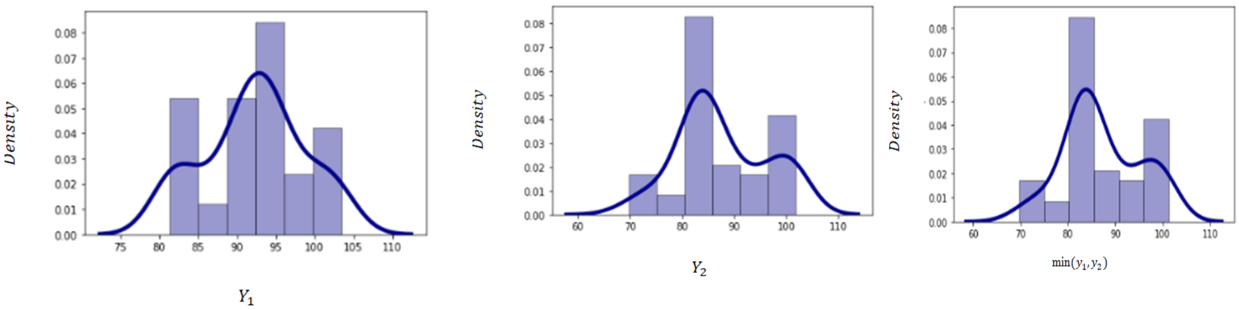


Figure (7): PDF for Y_1, Y_2 and $\min(y_1, y_2)$

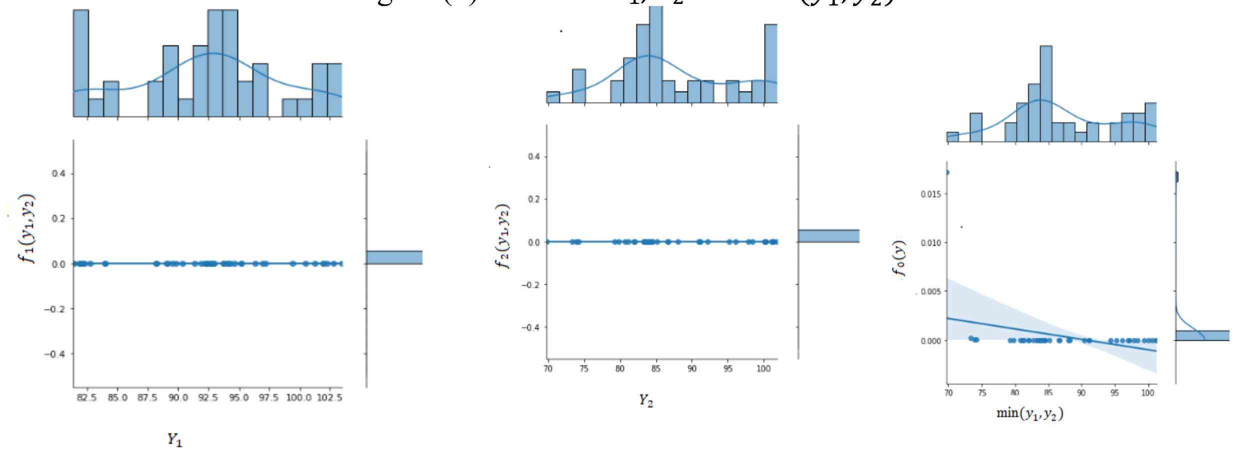


Figure (8): Y_1 , Y_2 and $\min(y_1 , y_2)$ with $f_1(y_1, y_2), f_2(y_1, y_2)$ and $f_0(y)$ respectively for second data set

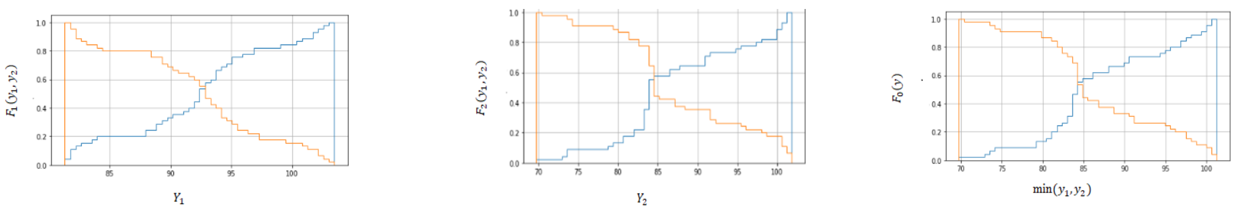


Figure (9): Y_1 , Y_2 and $\min(y_1, y_2)$ with $F(y_1, y_2)$and $F_0(y)$ respectively for first data set

From Figures (7-9), it is quite apparent that the marginals can be used to discuss this data. Therefore, the BOGEMW model may be used for this purpose. A summary of the values AIC, BIC, AICC and HQIC to compare BOGEMW, BGG and BEMW models is given in Table (4).

Table (4) l, AIC, CAIC, BIC and HQIC

Model	l	AIC	AICC	BIC	HQIC
BEMW	-244.637	497.275	498.275	504.501	499.969
BGG	-293.706	595.413	596.413	602.64	598.107
BOGEMW	-219.411	446.821	447.821	454.048	449.515

From Table (4) we can say that the BOGEMW distribution is the best distribution to explain the current data because it has the smallest value of AIC, CAIC, BIC and HQIC test, then the BOGEMW distribution fits better than the BGG and BEMW models.

8. Conclusions

In this paper, we have proposed a bivariate odd generalized exponential modified Weibull distribution whose marginals are odd generalized exponential modified Weibull distribution. We derive explicit expressions for some of its mathematical and statistical quantities including the ordinary and incomplete moments, cumulants, quantile and generating functions and probability weighted moments. The parameters have been estimated using maximum likelihood method based on complete data, and it was found that the maximum likelihood method performed quite well in estimating the parameters. The usefulness of the proposed model is illustrated by two real data sets and it was found that the new model provides a better fit than other sub models. We hope that the proposed model will attract wider applications in areas such as survival and lifetime data, meteorology, hydrology, engineering and others.

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توزيع وايبل الأسي الثنائي المعمم الفردي المعدل الجديد

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الملخص

قدم هذا البحث توزيع وايبل الأسي الثنائي المعمم الفردي المعدل الجديد. هذا التوزيع المقترح هو حالة عامة لبعض التوزيعات الثنائية الأخرى مثل توزيع وايبل الثنائي، توزيع وايبل المعمم الثنائي، توزيع وايبل الأسي الثنائي، التوزيع الأسي المعمم الفردي الثنائي وتوزيع وايبل المعدل الثنائي. اعتمد التوزيع المقترح علي طريقة مارشال وأولكين (١٩٦٧) إلي جانب أن الدالة الهامشية لهذا التوزيع الثنائي الجديد هي توزيع وايبل الأسي المعمم الفردي المعدل الذي قدمه عبد العال (٢٠١٦) وقدم البحث العديد من خصائص التوزيع الثنائي الجديد ولقد استخدمت طريقة الإمكان الأكبر لتقدير معلمات النموذج و اشتقاق مصفوفة المعلومات لفيشر ولقد تم تقييم التوزيع المقترح من خلال نماذج المحاكاة فضلا عن ذلك فقد تم تطبيق التوزيع المقترح علي نوعين من البيانات الواقعية والتي من خلالها أوضحت أهمية ومرونة التوزيع الجديد بالمقارنة ببعض التوزيعات الثنائية الأخرى .

الكلمات المفتاحية: التوزيعات الثنائية، تقدير دالة الإمكان الاكبر، توزيع وايبل المعدل، التوزيع الأسي المعمم الفردي، المحاكاة.